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# DYNAMICS OF INHOMOGENEOUS CHAINS OF COUPLED QUADRATIC MAPS

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*A new effective local analysis method is elaborated for coupled map dynamics. In contrast to the previously suggested methods, it allows visually investigating the evolution of synchronization and complex-behavior domains for a distributed medium described by a set of maps. The efficiency of the method is demonstrated with examples of ring and flow models of diffusively coupled quadratic maps. An analysis of a ring chain in the presence of space defects reveals some new global-behavior phenomena.*

**Keywords:** distributed media, space–time chaos, coupled map lattices

## 1. Introduction

Investigating lattices of coupled maps as models of distributed media has recently aroused great interest (see, e.g., [1]–[5]). In this approach, the lattice elements are the points of the physical medium, and the character of the coupling between them determines the interaction in accordance with the basic physical principles. The main problems among those arising in this case are replacing the continuous physical space with an adequate discrete analogue and choosing the correct coupling. But even with a favorable choice, the resulting lattice model rarely describes the abundance of phenomena observed in distributed systems. Therefore, one often resorts to considering these systems with an element evolution and coupling such that under the variation of the control parameters, they demonstrate a broad spectrum of phenomena characteristic of a distributed media [1], [2].

Lattice systems appear not only in considering distributed media but also in describing processes developing in systems having an essentially discrete structure in both space and time. Many problems in synchronization theory for radio generators, in biology, and in medicine and also the study of the behavior of cellular automata and neural networks lead to such models [4].

Depending on the dimensionality of the original modeled system, coupled-map lattice systems can be one-, two-, or three-dimensional. In this paper, we study only one-dimensional lattices, i.e., linear chains of the type of a set of elements interacting according to a definite law.

In constructing coupled-map lattices, we must first choose the form of the map determining the time evolution in each element. This choice of the map defines the local phase space $X$ on which the map $f\colon X \to X$ acts,

$$x \mapsto f(x), \quad x \in X. \tag{1}$$

The phase space of the entire lattice is the direct product of all local phase spaces of the individual elements. In the one-dimensional case, all elements can be arranged on the line and enumerated with a single index. Then the phase space of such a chain including $N$ elements can be written as $\mathcal{X} = \bigotimes_{i=1}^{N} X_i$. The state of the chain is therefore determined by the vector $\xi = (x_1, \dots, x_N) \in \mathcal{X}$.

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Another important problem in constructing map lattices is defining the coupling between the elements. Clearly, the coupling must specify the effect of the values of the elements of the entire lattice on the subsequent values of an individual element. In the one-dimensional situation, i.e., in the case of chains, the coupling can be defined by a map

$$x_i \mapsto g(x_1, \ldots, x_N), \quad i = 1, \ldots, N. \tag{2}$$

Hence, the dynamics of a coupled-map chain can be represented as the compositions of two maps (1) and (2). In this case, the value of the $i$th element varies during one time step according to the law

$$x_i \mapsto g\big(f(x_1), \ldots, f(x_N)\big), \quad i = 1, \ldots, N, \tag{3}$$

and the behavior of the entire chain decomposes into temporal and spatial parts determined by the respective map $f(x)$ and transformation $g(x_1, \ldots, x_N)$.

Although most of the results relating to coupled maps are obtained for this decomposition, it would be entirely incorrect to assert that this constraint is general. For example, in the temporal discretization of some systems of coupled ordinary differential equations and partial differential equations describing a real distributed medium, map systems appear whose evolution is defined by the transformation

$$x_i \mapsto f(x_i) + \bar{g}(x_1, \ldots, x_N), \quad i = 1, \ldots, N, \tag{4}$$

where, as before, the function $f(x)$ determines the temporal dynamics and the transformation $g(x_1, \ldots, x_N)$ defines the variation of the value of the element $x_i$ depending on the current state of the system. In this approach, the temporal and spatial transformations act simultaneously.

In the literature devoted to this topic, the most popular systems are one-dimensional chains of *diffusively coupled maps* [1]–[5]. A diffusive coupling means that the states of an element depend on only the values of their nearest neighbors. In this situation, the dynamics of a chain of type (3) is determined by a map in the form

$$x_i \mapsto f(x_i, a) + \frac{\varepsilon}{2}\big(f(x_{i-1}, a) - 2f(x_i, a) + f(x_{i+1}, a)\big), \tag{5}$$

where $\varepsilon$ is the diffusion coefficient. In some works(see, e.g., [1]–[4] and the references therein), the function $f(x, a)$ was taken in the form of a quadratic map $f(x, a) = 1 - ax^2$. Depending on the parameter values $a \in [0, 2]$ and $\varepsilon \in [0, 1]$, this transformation can manifest a very broad spectrum of types of global behavior ranging from synchronization of all elements and periodic temporal and spatial dynamics to space–time chaos.

Along with chains of diffusively coupled maps (5), the behavior of maps of the form

$$x_i \mapsto f(x_i, a) + \frac{\varepsilon}{2}(x_{i-1} - 2x_i + x_{i+1}) \tag{6}$$

is often investigated, where $x \in [0, 1]$, $a$ is the nonlinearity parameter, and the coefficient $\varepsilon$ characterizes the interaction between the elements. This choice of the coupling means that the spatial and temporal actions under map (6) occur simultaneously as in the case of transformation (4).

Apart from the indicated types of interaction, the so-called flow model approximating a free fluid flow is sometimes considered. In this case, the chain is a system of unilaterally coupled maps [5]–[8],

$$x_i \mapsto f(x_i, a) + \varepsilon\big(f(x_{i-1}, a) - f(x_i, a)\big), \tag{7}$$

where $f(x,a)$ defines the evolution of every chain element. Depending on the parameter values $a$ and $\varepsilon$, this transformation of the function $f(x,a) = 1 - ax^2$ demonstrates a broad spectrum of types of space–time behavior such as space chaos and periodicity in time, spatial (and temporal) periodic and quasiperiodic structures, and also some definite space–time forms. These forms include different kinds of formations ranging from individual structures surrounded by chaos to developed space–time chaoticity.

It is clear that for any chain, it is necessary to indicate its behavior on the boundary. The most frequently studied systems are coupled maps with free or periodic boundary conditions. Free boundary conditions are defined by the relations $x_0 = x_{N+1} \equiv 0$ (or $x_0 \equiv 0$ for systems of form (7)). For periodic boundary conditions, it is necessary to set $x_k = x_{N+k}$, $k = 1, \ldots, N$.

If the map parameter values differ in a lattice, then the related system is no longer homogeneous, and investigating it becomes a much more complicated problem [9]. In practically all works devoted to studying diffusively interacting maps, homogeneous lattices are investigation (as a rule, numerically). But from the physical standpoint, the homogeneity of a space (understood as the identity of all elements or the equality of parameter values for the elements forming the lattice in the case under consideration) is an idealization introduced to simplify the analysis. It is therefore very interesting to discover what qualitative changes in the system dynamics are caused by the appearance of inhomogeneities. There can be quite various forms of inhomogeneity ranging from individual defects to periodic inhomogeneity throughout the space. In this paper, defects are understood as a difference in parameter values for some of the elements forming the lattice. Some examples of these inhomogeneous distributed systems are considered in Sec. 4.

From the mathematical viewpoint, coupled-map lattices with a finite number of elements are dynamic systems with a finite number of degrees of freedom. For these systems, an apparatus for analytic and numerical investigation is developed. But if $N$ is very large, then the calculation of the main characteristics of the dynamic system is either very cumbersome or impossible in principle. Moreover, many of the parameters that must be calculated for dynamic systems (such as the metric entropy, the spectrum of Lyapunov exponents, the rate of decrease of temporal correlations, etc.) reflect the *asymptotic behavior* of the system as $t \to \infty$ and throughout the space as a whole [10], [11]. But knowing these parameters for coupled-map systems is essentially uninteresting. It is more important to find the characteristics determining the local properties of these systems and their evolution in time. Some authors suggested various means for observing the evolution of lattice models ranging from visual analysis methods to calculation of the local Lyapunov exponents (see, e.g., [12], [13] and the references therein).

In this paper, we develop a new analysis method for coupled-map lattices that permits determining the behavior of the individual elements and the dynamics of the system as a whole. In contrast to the other methods, is allows investigating the local behavior of individual elements of a distributed system and the evolution of the global dynamics of the entire phase space. The method is based on decomposing the entire time interval into short subintervals on which the degree of nonperiodicity of the trajectories of all lattice elements is analyzed. This makes it possible to determine the synchronization domains, their temporal period, their transformation in time, and their break under spatial "chaotization." This paper is a continuation of the investigations of inhomogeneous coupled map lattices initiated in [9].

## 2. Local criterion for coupled-map dynamics

Although the present paper is devoted to studying one-dimensional coupled-map lattices, the method presented in this section can be used to investigate the multidimensional case as well. By construction, it permits analyzing the state of an element in the space of the distributed system on a small time interval based on knowing the time sequence at the given point.

The values of each lattice element at consecutive instants in the evolution process form a series

$$x_i^1, x_i^2, x_i^3, \dots, x_i^n, \dots, \tag{8}$$

where $i$ is the index of the element and $n$ is the discrete time. To reveal the periodicity of the series and determine its variation in time, it is obviously necessary to take an arbitrary segment of the series and to try to determine to what extent it is close to a periodic one. Decomposing the entire series into short segments of the same length, we admit a certain coarsening with respect to time. Analyzing the periodicity of the series in each of these segments and expressing it by a number, we can trace the evolution of the given characteristic in the transition from one segment to another.

We now describe this procedure in more detail. Let $T$ be the number of elements in the series, i.e., let the series have the form $x_1, x_2, \dots, x_T$ (for brevity, we omit the index $i$). We decompose it into identical time intervals $\tau$ and choose $T$ and $\tau$ such that $T = k\tau$, where $k$ is a positive integer. Next, we test each of the series segments of length $\tau$ for periodicity. For this, we perform the following procedure for each segment $x_1, \dots, x_\tau$ of the series.

1. We compare it with a specially constructed series $x_1, x_1, x_1, \dots, x_1$ such that all its elements are equal to $x_1$ and then express the result by a number $\lambda_1$. The procedure for calculating $\lambda_1$ is presented below. At the moment, it is important that the number $\lambda_1$ should have the property that the smaller $\lambda_1$ is, the closer the series segment in question is to the specially constructed series $x_1, x_1, x_1, \dots, x_1$.
2. The given series is compared with the series $x_1, x_2, x_1, x_2, \dots, x_1, x_2, \dots$ consisting of periodic subsequences of period two. The comparison result is expressed by a number $\lambda_2$.
3. A similar operation is performed for the series composed of three elements $x_1, x_2, x_3$, four elements $x_1, x_2, x_3, x_4$, etc., up to $x_1, x_2, \dots, x_{\tau/2}$. The comparison results are expressed by numbers $\lambda_3, \lambda_4, \dots, \lambda_{\tau/2}$, which again reflect some degree of deviation of series (8) under study from the corresponding model series composed of individual consecutive elements.
4. We find the parameter

$$\lambda \equiv \lambda_{\min} = \min\{\lambda_1, \lambda_2, \dots, \lambda_{\tau/2}\}. \tag{9}$$

The comparative figure $\lambda_p$ is calculated in the general form for each $p = 1, \dots, \tau/2$ by the formula

$$\lambda_p = \sqrt{\frac{\sum_{i+1}^{\tau} \bigl(x_i - x_{i \pmod{(p+1)}}\bigr)^2}{\tau - p}}\,. \tag{10}$$

It can be seen easily that $\lambda_p$ is the mean square deviation of the elements of the residual $x_{p+1}, \dots, x_\tau$ of the original series from the model periodic series $x_1, x_2, \dots, x_p, x_1, x_2, \dots, x_p, \dots$. According to (10), the inequality $\lambda_p \geq 0$ holds for each value of $p$. It is clear that if the relation $\lambda_{\min} = \lambda_p = 0$ holds for some $p$, then the series $x_1, \dots, x_\tau$ is periodic with a period multiple of $p$. To determine the period of the original series exactly, we must stop the procedure for calculating the figures $\lambda_p$ at the first value of $p$ such that $\lambda_p \simeq 0$. But if the calculation process results in $\lambda = \min\{\lambda_1, \lambda_2, \dots, \lambda_{\tau/2}\} > 0$, then we can definitely assert that the series $x_1, \dots, x_\tau$ is not periodic. The value of $p$ for which the given minimum is attained can be called a period that approximates the given aperiodic series. Calculating the value of $\lambda$ for the individual segments of length $\tau$, we can trace their dynamics throughout the entire interval of length $T$. Calculating $\lambda$ and the periods on which the minimum is attained for all points in the chain, we can effectively reveal the synchronization domains in the space of the coupled-map system in question and study the dynamics of the individual elements.

It is easy to understand that, rigorously speaking, the suggested analysis method for coupled-map lattices is not exact. We must first choose an adequate value of $\tau$, i.e.., the length of the analyzed segment. The most suitable value of $\tau$ can be found by calculating the sets of $\lambda_p$ for different values of $\tau$. Another disadvantage is that the figure $\lambda_{\min}$ is positive for elements whose dynamics is quasiperiodic. Nevertheless, it is sufficiently close to zero because quasiperiodic motion can be approximated arbitrarily accurately by periodic motion. But because we are interested in the qualitative pattern of the distribution of $\lambda_{\min}$ rather than in concrete parameter values, this disadvantage is not very essential.

## 3. Examples of homogeneous systems

We demonstrate the efficiency of the suggested method with several well-known examples.

**3.1. Flow model.** We investigate different types of dynamics for flow model (7). As already mentioned, chain (7) has a broad spectrum of different behavior modes depending on the nonlinearity degree determined by the parameter $a$ and by the values of the diffusion coefficient $\varepsilon$. Moreover, the simplest types of limiting behavior can be calculated analytically. We consider this in more detail.

The model under investigation can be expressed in the form of an iterative relation,

$$x_{n+1}^{i} = f(x_n^i, a) + \varepsilon\bigl(f(x_n^{i-1}, a) - f(x_n^i, a)\bigr), \quad i = 1, \dots, N, \tag{11}$$

where $n$ is the discrete time and $i$ is the spatial coordinate. Let $f(x, a) = 1 - ax^2$ and $x^0 \equiv 0$ be taken as the boundary conditions, although their choice does not affect the qualitative investigation results.

The general state of system (11) at the instant $n$ is obviously defined by the $N$-dimensional vector $\xi_n = (x_n^1, x_n^2, \dots, x_n^N)$. Consequently, the dynamics of the entire chain is expressed by some transformation $F\colon \mathcal{X} \to \mathcal{X}$, $\xi_{n+1} = F(\xi_n)$, of the phase space $\mathcal{X}$ into itself. It can be clearly seen from the construction of the model that $F = g \circ f$ (see (1) and (2)). Formally (see above), lattice (11) is a discrete-time dynamic system with $N$ degrees of freedom. Therefore, studying stationary and periodic states reduces to analyzing the spectrum of eigenvalues of the Jacobi matrix $\mathrm{D}F$ and its powers $\mathrm{D}F^{(p)}$, where $p$ is the length of the period under investigation. We write the expression for the matrix $\mathrm{D}F$,

$$\mathrm{D}F(\xi) = \begin{pmatrix} -(1-\varepsilon)2ax^1 & 0 & 0 & \cdots & 0 \\ \varepsilon & -(1-\varepsilon)2ax^2 & 0 & \cdots & 0 \\ 0 & \varepsilon & -(1-\varepsilon)2ax^3 & \cdots & 0 \\ 0 & 0 & \varepsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -(1-\varepsilon)2ax^N \end{pmatrix}. \tag{12}$$

We note that $\mathrm{D}F$ is a triangular matrix. Consequently, its powers $\mathrm{D}F^{(p)}$ are also triangular matrices.

We now calculate the homogeneous stationary states of model (7). The homogeneity means that the relation $x_n^i = x_n^*$ holds for any $i = 1, \dots, N$, and the stationarity expresses the time independence of the homogeneous state, i.e., $x_n^i = x_n^* \equiv x^*$. Hence, the values of $x^*$ must satisfy the condition

$$x^* = f(x^*, a) + \varepsilon\bigl(f(x^*, a) - f(x^*, a)\bigr) = f(x^*, a). \tag{13}$$

In other words, the $x^*$ are fixed points of the map defined by $f(x, a)$,

$$x_{1,2}^* = \frac{-1 \pm \sqrt{1 + 4a}}{2a}. \tag{14}$$

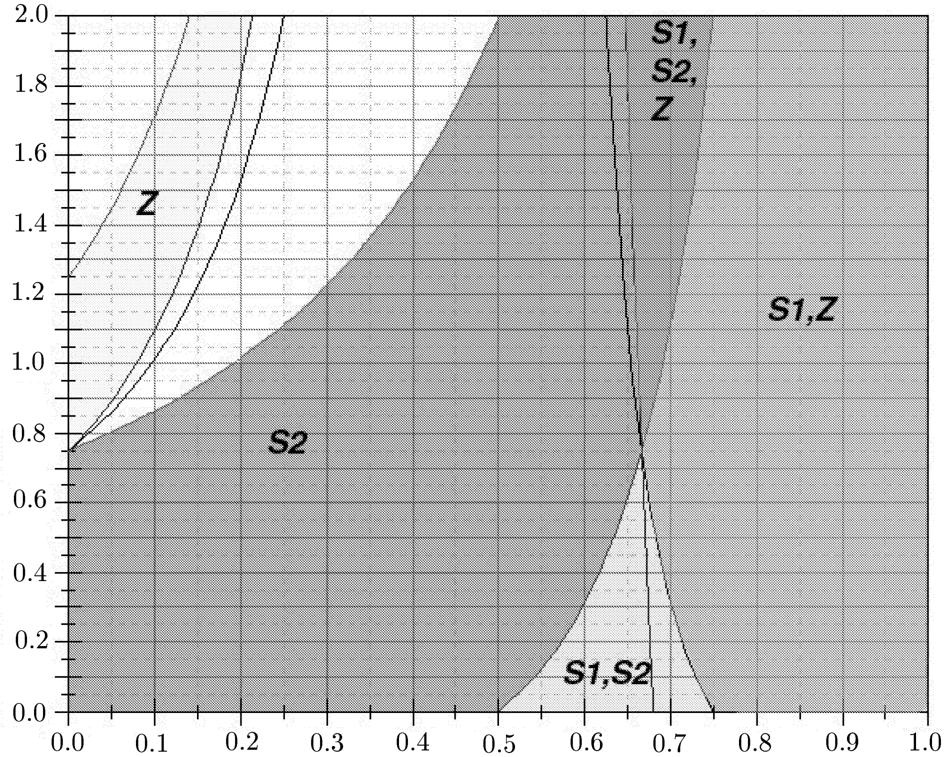


**Fig. 1.** Existence and stability domains for the stationary states $x_1^*$ and $x_2^*$ and for the zigzag behavior of chain (7). Here, $S1$ and $S2$ are the respective existence domains of the stable points $x_1^*$ and $x_2^*$, and $Z$ is the existence domain of zigzag solutions (18).

Thus, system (7) has two homogeneous states, $\xi_1 = (x_1^*, x_1^*, \ldots, x_1^*)$ and $\xi_2 = (x_2^*, x_2^*, \ldots, x_2^*)$. Their stability is determined by the magnitudes of the eigenvalues of the matrices $\mathrm{D}F(\xi_1)$ and $\mathrm{D}F(\xi_2)$. Because $\mathrm{D}F(\xi)$ is a triangular matrix, we can use (12) to find the eigenvalues $\lambda_i \equiv \lambda_1 = -(1-\varepsilon)2ax_1^*$ and $\lambda_i \equiv \lambda_1 = -(1-\varepsilon)2ax_2^*$ for the respective matrices $\xi_1$ and $\xi_2$. The states $\xi_1$ and $\xi_2$ are stable if $|\lambda_{1,2}| \le 1$, i.e.,

$$-1 < -(1-\varepsilon)2ax_1^* < 1, \qquad -1 < -(1-\varepsilon)2ax_2^* < 1. \tag{15}$$

Substituting $x_1^*$ and $x_2^*$ results in a relation between the parameters $a$ and $\varepsilon$,

$$\begin{aligned} 1 - \frac{1}{1+\sqrt{1+4a}} < \varepsilon < 1 + \frac{1}{1+\sqrt{1+4a}} \quad &\text{for } x_1^*, \\ 1 - \frac{1}{\sqrt{1+4a}-1} < \varepsilon < 1 + \frac{1}{\sqrt{1+4a}-1} \quad &\text{for } x_2^*. \end{aligned} \tag{16}$$

Figure 1 demonstrates the calculation results for domains (16) with $\varepsilon \in [0,1]$ and $a \in [0,2]$. The domains marked by $S1$ and $S2$ correspond to the stable points $x_1^*$ and $x_2^*$.

Coupled-map chain (7) has another remarkable type of periodic behavior in a wide range of the parameters $(a, \varepsilon)$. Namely, in the parameter range $a \in [1.6, 2.0]$, the dynamics of the entire system manifests the so-called zigzag motion in which the behavior of all elements (or of the majority of the elements) has a cyclicity of period two in both space and time. In this behavior, the trajectories of the individual points are two-periodic, but the neighboring elements are in antiphasic oscillations.

We investigate such a state analytically. The period two of each element is determined by the two values $x_1$ and $x_2$. The cyclicity in space and time is expressed by the relations

$$
\begin{aligned}
x_1 &= (1-\varepsilon)f(x_2) + \varepsilon f(x_1),\\
x_2 &= (1-\varepsilon)f(x_1) + \varepsilon f(x_2).
\end{aligned}
\tag{17}
$$

Solving Eqs. (17) and eliminating the homogeneous states $x_1 = x_2$, we obtain

$$
x_{1,2} = \frac{1 \pm \sqrt{4(1-2\varepsilon)^2 a + 4\varepsilon - 3}}{2(1-2\varepsilon)a}. \tag{18}
$$

The stability of the resulting periodic state can be investigated by analyzing the eigenvalues of the matrix product $\mathrm{D}F(\hat{\xi}_1)\mathrm{D}F(\hat{\xi}_2)$, where $\mathrm{D}F$ is given by (12), $\hat{\xi}_1 = (x_1, x_2, \dots, x_1, x_2, \dots)$, and $\hat{\xi}_2 = (x_2, x_1, \dots, x_2, x_1, \dots)$. It is clear that the product of the triangular matrices $\mathrm{D}F(\hat{\xi}_1)$ and $\mathrm{D}F(\hat{\xi}_2)$ is the triangular matrix $\mathrm{D}F^2$. The elements on the diagonal of $\mathrm{D}F^2$ are the products of the diagonal elements of $\mathrm{D}F(\hat{\xi}_1)$ and $\mathrm{D}F(\hat{\xi}_2)$. Therefore, the eigenvalues of $\mathrm{D}F^2$ can be written as

$$
\lambda_i \equiv \lambda = (1-\varepsilon)^2 f'(x_1) f'(x_2) = (1-\varepsilon)^2 4a^2 x_1 x_2,
$$

whence, in view of (18), it follows that

$$
\lambda = \frac{4(1-\varepsilon)^2}{(1-2\varepsilon)^2} - 4(1-\varepsilon)^2 a.
$$

Finally, to construct the domain where the zigzag behavior of $(x_1, x_2)$ is observed, we must take the system of inequalities

$$
4(1-2\varepsilon)^2 a + 4\varepsilon - 3 \geq 0, \qquad -1 < \lambda < 1 \tag{19}
$$

into account. System (19) reflects the existence and stability conditions for this behavior. The calculation with inclusion of the additional constraints on the parameters ($a \in [0,2]$ and $\varepsilon \in [0,1]$) leads to the relations

$$
\begin{aligned}
&\frac{4(1-\varepsilon)^3 - (1-2\varepsilon)^2}{4(1-\varepsilon)^2(1-2\varepsilon)^2} < a < \frac{4(1-\varepsilon)^3 + (1-2\varepsilon)^2}{4(1-\varepsilon)^2(1-2\varepsilon)^2},\\
&a \geq \frac{3-4\varepsilon}{4(1-2\varepsilon)^2}.
\end{aligned}
\tag{20}
$$

The resulting construction of domain (20) is shown in Fig. 1, where it is denoted by $Z$. As follows from Fig. 1, the domain $Z$ can be overlapped with the existence domains $S1$ and $S2$ for stable points. This means that the limit state of chain (11) depends on the initial distribution $\{x_0^i\}_{i=1}^N$ for some fixed values of the parameters $a$ and $\varepsilon$ belonging to the intersection domains. The dynamics of these chains is said to be multistable.

The boundary condition $x^0 \equiv 0$ has not been taken into account in all calculations relating to stationary and periodic states and their stability (see (13), (15), (17), and (19)). In other words, all the results are obtained in the absence of a boundary, i.e., on the condition that $N \to \infty$. But these results adequately reflect the qualitative pattern of the dynamics of chain (7). Moreover, in this case, the boundary condition can be regarded as some constant external perturbation of the coupled-maps that corrects the dynamics of the limit state.

The system behavior at large values of the dimensionality $N$ is essentially metastable. This manifests itself in the trajectory behavior strongly depending on the domain from which the initial data $(x_0^1, x_0^2, \dots, x_0^N)$ are chosen. The above analysis of stable solutions in no way reflects the related attraction domains. Therefore, the determination of these states may encounter some difficulties even in an exact calculation (with regard to the boundary conditions).

Clearly, the variety of the possible behavior modes of system (7) is not at all confined to solutions (14) and (18). To demonstrate the variety of the dynamics for chain (11) with $f(x,a) = 1 - ax^2$, we present the numerical experiment results for the given model based on the criterion described in Sec. 2.

**1.** $a = 1.7$, $\varepsilon = 0.45$. In the chain, we observe temporal periodicity and doubling of the period in element-to-element motion from left to right. But the values of the elements $x^i$ are distributed in space chaotically, which is demonstrated in Fig. 2. The smaller the values of $\lambda$ and $p$ are, the darker the corresponding point is in the graph. The value $\lambda = 0$ corresponds to the darkest points in the diagram for $\lambda$. It is easy to see that the values of $\lambda$ for all elements and at all instants are zero, and the lengths of the periods in this case are doubled from left to right up to $p = 32$. To analyze the spatial pattern, the values of $x^i$ at some consecutive instants are also shown in Fig. 2. It is intuitively clear that this chain shows spatial chaos. This can be confirmed more rigorously based on an analysis using a specially constructed map (see [5]). A more detailed investigation of this state showed that the value of the maximum period up to which the doubling continues depends only on $a$ and $\varepsilon$, and the index of the element after which no further doubling occurs increases as $N$ increases.

**2.** $a = 1.7$, $\varepsilon = 0.11595$. For the given values of the degree of nonlinearity and the coupling force, the authors of [8] revealed a type of behavior under which the majority of the elements are in zigzag motion, whereas some equidistant individual points are in chaotic motion. In this case, the distance between the chaotic defects depends logarithmically on the difference $\varepsilon - \varepsilon_c$, where $\varepsilon_c \approx 0.11525$ is some critical value. The described phenomenon is reflected in Fig. 3. The chaotic defects are expressed by clear "flickering" lines, and the distance between them can be calculated easily.

**3.2. Ring chain of quadratic maps.** We consider a one-dimensional lattice of form (5),

$$x_{n+1}^i \mapsto f(x_n^i, a) + \frac{1}{2}\varepsilon\big(f(x_n^{i-1}, a) - 2f(x_n^i, a) + f(x_n^{i+1}, a)\big), \tag{21}$$

where $f(x,a) = 1 - ax^2$. System (21) was intensively investigated earlier in [1]–[3], where it was shown that this system has an extensive variety of behavior modes ranging from synchronization of spatial structures to fully developed turbulence. As in the case of chain (7), the dynamics of system (21) essentially depend on the values of the parameters $a$ and $\varepsilon$. We investigate some of the types of behavior for this chain using an analysis of point trajectories by calculating the figures $\lambda$ (see Sec. 2) and the period values $p$ in different time intervals.

**1.** $a = 1.44$, $\varepsilon = 0.1$. Depending on the initial conditions, time-stable domains with regular and irregular dynamics are formed in space. Despite the diffusive coupling between the elements, the pattern of the system behavior is stable. This behavior of system (21) can be called a state of "frozen" structures. A typical pattern of these structures is presented in Fig. 4. This figure demonstrates a stable pattern of variation of the figure $\lambda$ and the related period values $p$.

**2.** $a = 1.88$, $\varepsilon = 0.3$. An increase of the degree of nonlinearity determined by the parameter $a$ in an individual map $f(x,a)$ leads to the destruction of all spatial synchronization domains. In this case, the dynamic behavior of the entire chain, as well as that of the individual elements, rapidly changes in both time and space [1]. This is well reflected by the irregularity parameter $\lambda$, whose behavior is demonstrated

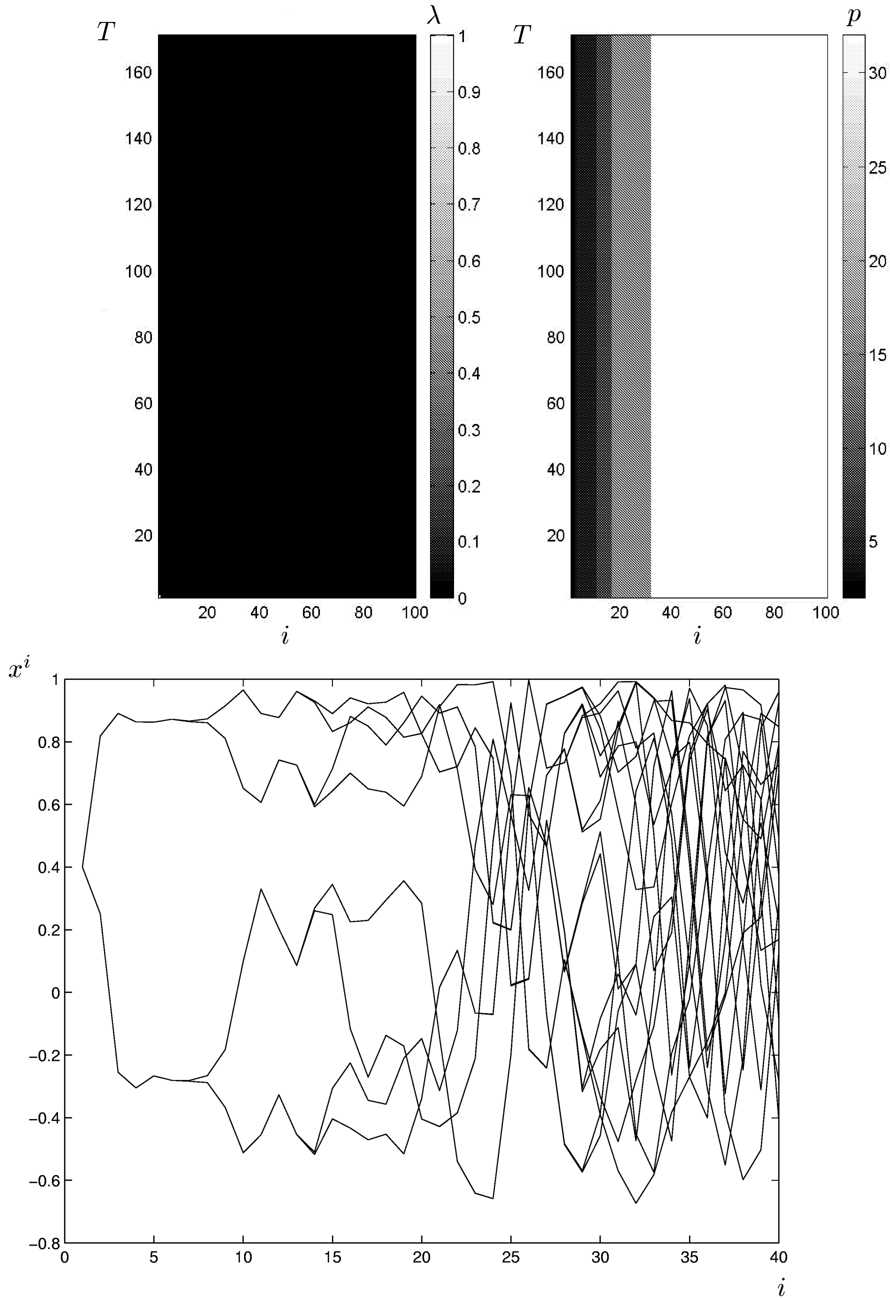


**Fig. 2.** Dynamics of system (11) for the parameter values $a = 1.7$ and $\varepsilon = 0.45$. Shown on the left and right are the values of the figure $\lambda$ depending on the index of the element (the horizontal axis) and on time in the scale $\tau$ (the vertical axis) and the corresponding values of the periods $p$. At the bottom, the distribution of $x^i$ at some consecutive instants is shown.

in Fig. 5. A similar pattern is also observed for the distribution of periods $p$ of individual elements. As a confirmation of this property, the same figure shows the distribution of the values of $x_n^i$ at several consecutive instants $n$. This behavior of coupled-map chains is usually called space–time chaos or, using the terminology of the theory of nonequilibrium media, fully developed turbulence [1].

**3.** $a = 1.8$, $\varepsilon = 0.3$. In this, case, the domains of regular and chaotic dynamics alternate for randomly chosen initial conditions. This behavior is observed for a rather long time. But almost all chain elements

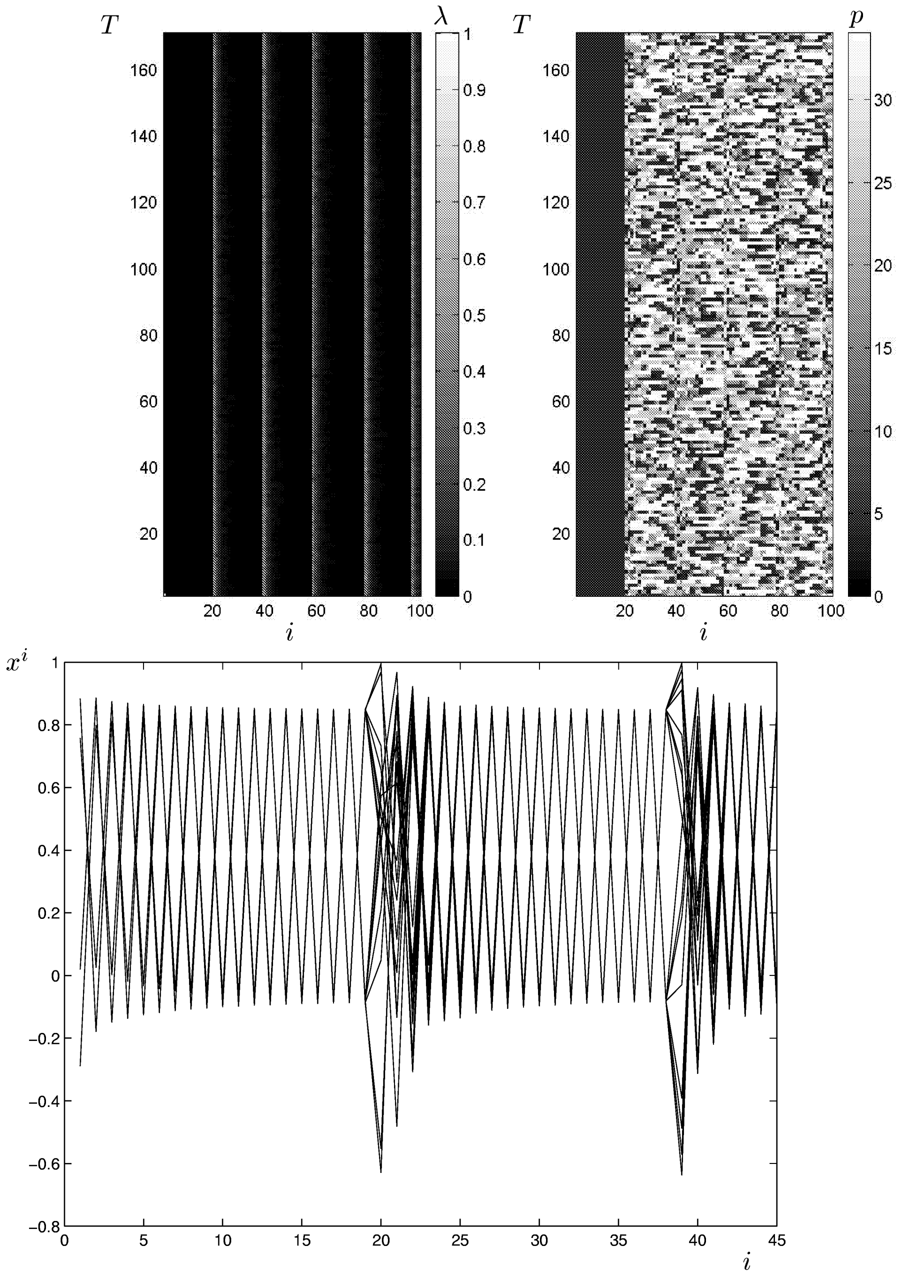


**Fig. 3.** The same as in Fig. 2 for $a = 1.7$ and $\varepsilon = 0.11595$.

ultimately start behaving regularly. An analysis shows that the chain tends to decompose into two-element cells in each of which the dynamics has the period two. This is demonstrated in Fig. 6, which shows the behavior of system (21) after 1 527 000 preliminary iterations.

The general analysis of systems (7) and (21) in the above cases and for some other parameter values shows that the elaborated visualization method for the dynamics of a distributed medium approximated by coupled-map lattices is an effective means of investigating space–time evolution.

## 4. An analysis of inhomogeneous one-dimensional coupled-map chains

Investigation of nonlinear distributed systems shows that the presence of inhomogeneities (defects)

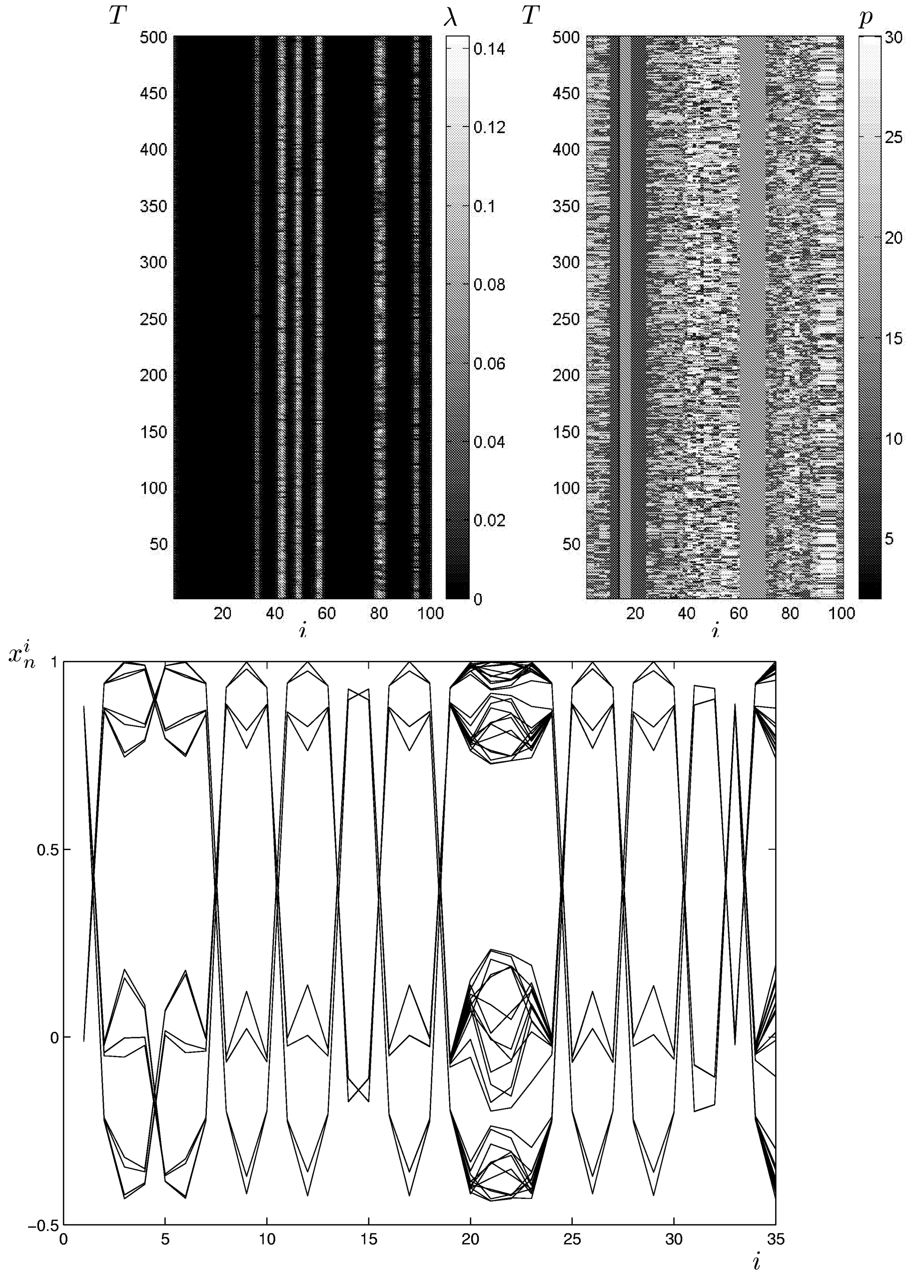


**Fig. 4.** Dynamics of ring chain (21) for the parameter values $a = 1.44$ and $\varepsilon = 0.1$.

mentioned in the introduction can substantially change the character of the system dynamics (see [9] and [14]). From the formal standpoint, the presence of at least a single element with a defect in a coupled-map lattice increases the number of control parameters. As a result, the dimensionality of the parameter space of the lattice regarded as a dynamic system increases, which in turn extends the set of possible behavior modes because of the nonlinearity of the evolution law. It is clear that the investigation of these modes in such a situation also becomes more complicated.

We apply the elaborated method (see Sec. 2) to demonstrate the result of the presence of some defects in one-dimensional lattices of diffusively coupled maps.

We consider chain (21) of length $N = 100$ with $f(x, a) = 1 - ax^2$ in which all elements (except the

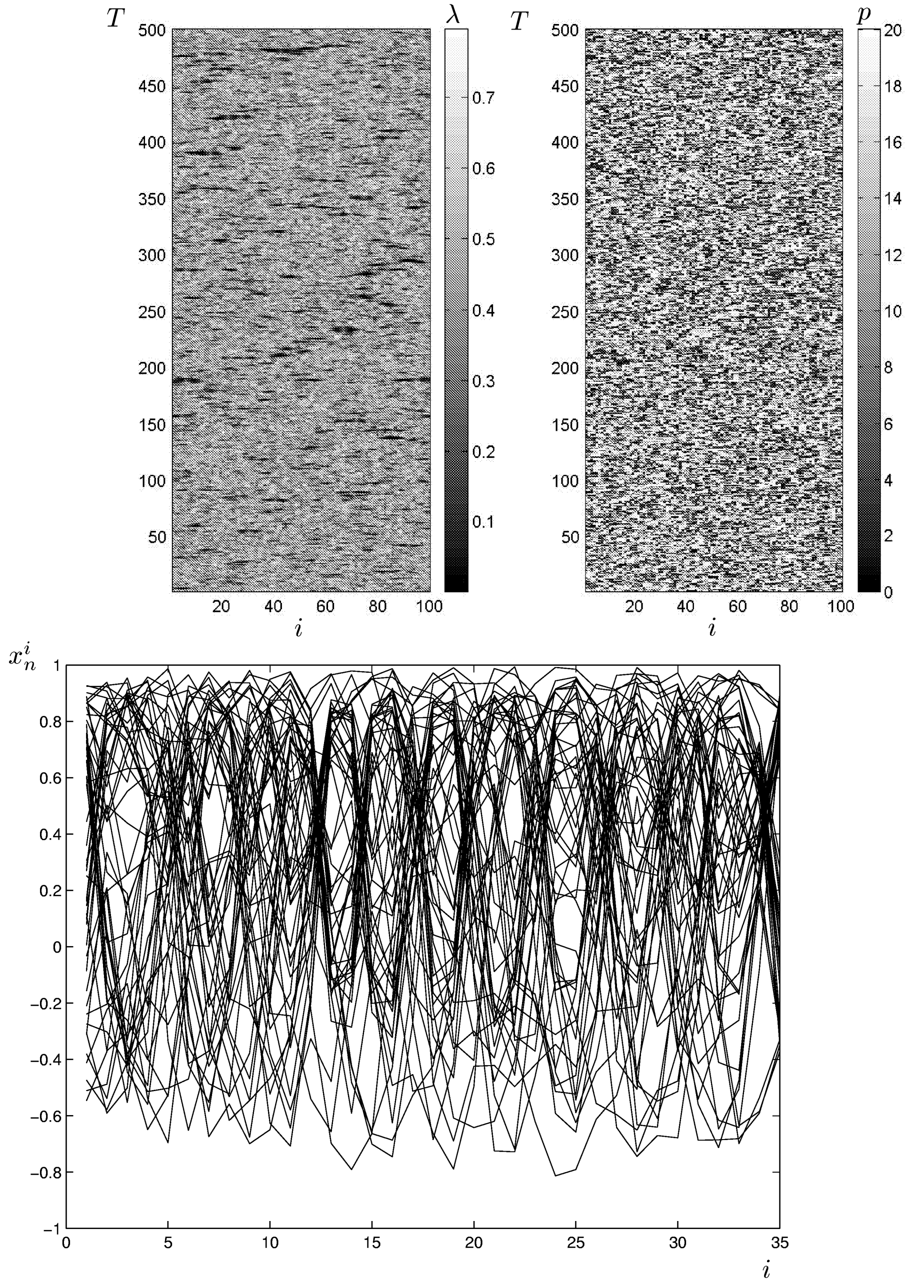


**Fig. 5.** The same as in Fig. 4 for the parameter values $a = 1.88$ and $\varepsilon = 0.3$.

central ones) have the same parameter value $a = a_1 = 1.44$ and the central elements (with the indices $i$ =47–53) have the different parameter value $a = a_2 = 1.97$. For $a = a_1 = 1.44$ and for a wide range of the values of $\varepsilon$, regular behavior in the form of small synchronized spatial structures is observed in *homogeneous* chain (21) (see above). In the case $a = a_2 = 1.97$, homogeneous system (21) manifests all properties of space–time chaos. Thus, conditionally, the defect consists in that several "chaotic" elements are impregnated in a chain with regular behavior. The results of analyzing 5000 iterations of such a chain after $10^5$ preliminary iterations are presented in Fig. 7. As is seen, the central elements are synchronized with the period two, and the chaos is moved to the other elements. But if $5 \cdot 10^6$ preliminary iterations are performed, then the chaos is localized in the elements with $a_2 = 1.97$ and in some neighboring elements on

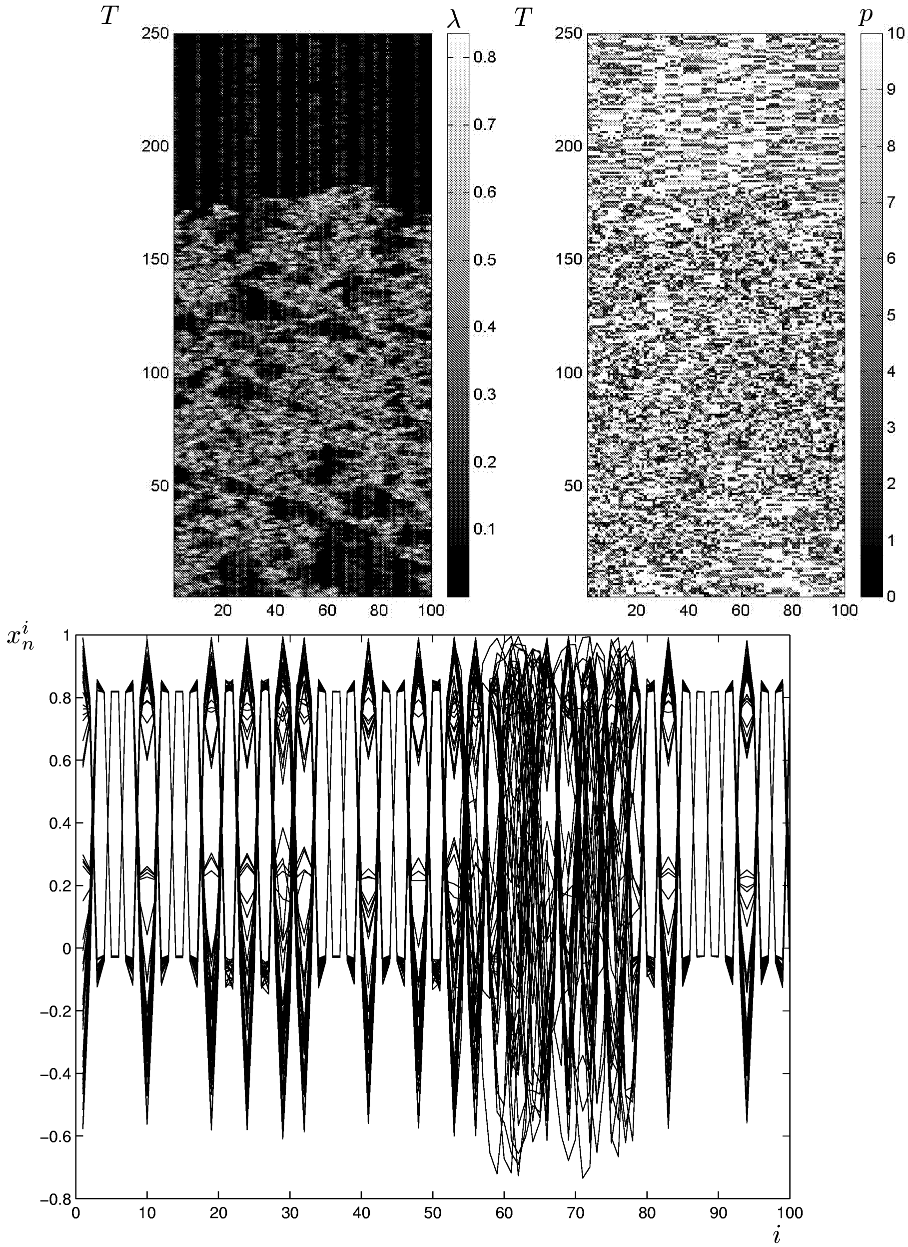


**Fig. 6.** The same as in Fig. 4 for the parameter values $a = 1.8$ and $\varepsilon = 0.3$.

the right and left. This shows that because of diffusion, the chaotic behavior propagates to elements with regular behavior.

We now consider a different case of inhomogeneous chain (21). We recall that system (21) with the parameter value $a = 1.8$ and small values of the diffusion coefficient $\varepsilon$ shows alternation of synchronization and also irregular behavior in both space and time (see Sec. 3.2). We construct a chain of form (21) with $N = 100$ such that 50 elements have the same value of the nonlinearity parameter ($a = a_1 = 1.8$) and the other 50 elements have the different value $a = a_2 = 1.97$,

$$x_{n+1}^i = \begin{cases} f(x_n^i, a_1) + \dfrac{\varepsilon}{2}\big(f(x_n^{i-1}, a_1) - 2f(x_n^i, a_1) + f(x_n^{i+1}, a_1)\big) & \text{for odd } n, \\ f(x_n^i, a_2) + \dfrac{\varepsilon}{2}\big(f(x_n^{i-1}, a_2) - 2f(x_n^i, a_2) + f(x_n^{i+1}, a_2)\big) & \text{for even } n. \end{cases} \tag{22}$$

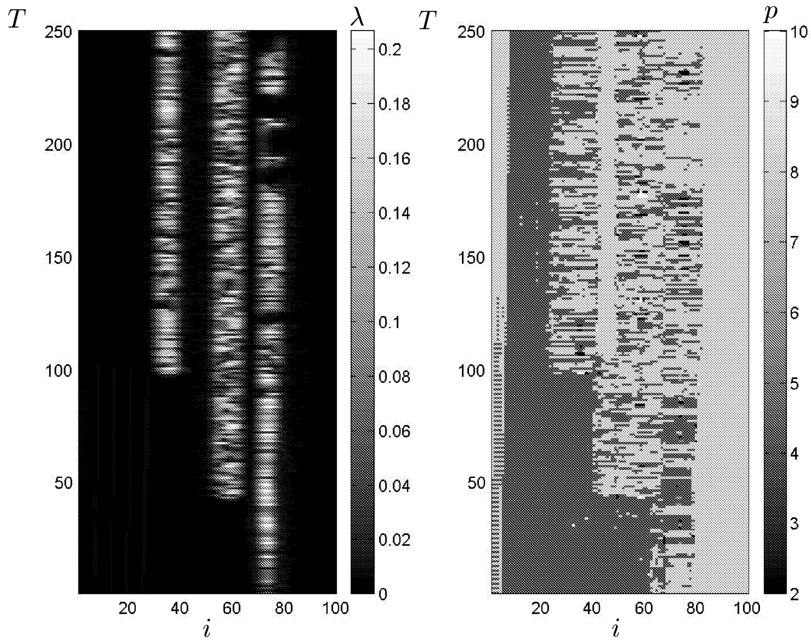


**Fig. 7.** Dynamics of chain (21) (after $10^5$ preliminary iterations) all of whose elements (except for the central ones) have the same parameter value $a = a_1 = 1.44$ and whose central elements (with the indices $i$ =47–53) have the different parameter value $a = a_2 = 1.97$. The diffusion coefficient is $\varepsilon = 0.7$.

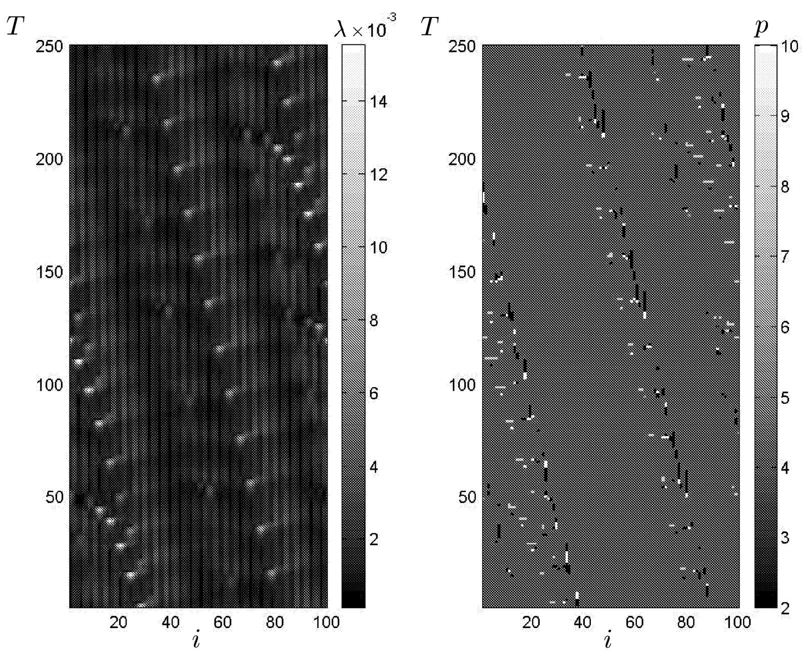


**Fig. 8.** Behavior of system (22) with alternation of the nonlinearity parameter in the elements after $5 \cdot 10^7$ preliminary iterations. The other parameter values are $a_1 = 1.8$, $a_2 = 1.97$, and $\varepsilon = 0.7$.

Thus, elements with different values of $a$ in system (22) alternate in space. We set the value 0.7 for the diffusion parameter $\varepsilon$. This inhomogeneous system manifests space–time chaos on relatively short time intervals $T \approx 10^4$ by analogy with the behavior of the homogeneous chain with $a = 1.97$. But

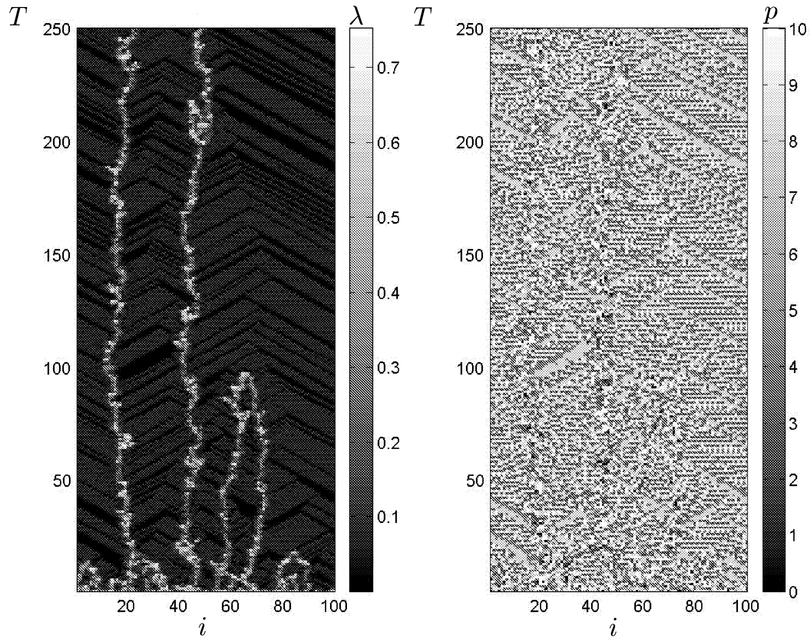


**Fig. 9.** Dynamics of ring chain (21) for the parameter values $a = a_1 = 1.8$ and $\varepsilon = 0.1$.

if an asymptotic analysis of this dynamics is performed (e.g., by performing $5 \cdot 10^7$ iterations), then an unexpected phenomenon can be revealed, namely, the entire chain is synchronized. This manifests itself in a regular pattern being observed in the space of values of the parameter $\lambda$ (see Fig. 8). Here, splashes of irregularity appear at equal time intervals and at the same distances from one another, i.e., it can be said that the chaos is in a kind of motion with a constant velocity along the chain. Figure 8 also demonstrates some small bent branchings from the straight lines. This means that the chaotic splashes move with a small acceleration rather than with a constant velocity. But they disappear after some time period. If this phenomenon is investigated thoroughly, then it can be found that the velocity of motion of the chaotic defects essentially depends on the values of $a_1$ and $a_2$ and also on the diffusion coefficient $\varepsilon$ but is independent of the distribution of the initial conditions. Only the direction of their motion depends on the initial conditions.

In conclusion, we investigate inhomogeneous ring chain (21) with a single defect. We let all chain elements have the same degree of nonlinearity ($a = a_1 = 1.8$) except that $a = a_2 = 1.99$ for the central element, and we choose the diffusion coefficient $\varepsilon = 0.1$. We first consider system (21) without defects. A typical dynamic pattern is show in Fig. 9. Most of the elements are synchronized during the entire evolution period, but, at the same time, there exist some small chaotic structures that wander in space. As is well seen from Fig. 9, they are mutually annihilated when they collide. Accordingly, they are also created in pairs. This behavior was compared to Brownian motion in [1] and [2].

We now add a defect with $a = a_2 = 1.97$ at the center. Investigations show that the presence of an element with a higher degree of nonlinearity leads to a similar pattern except that in the central element there always exists synchronization detuning and the number of wandering structures increases. Their dynamics completely repeat that of the structures in a homogeneous system with the only distinction that they can also disappear when colliding with the center ($i = 50$) and can be created at the center. Conditionally, the element with defect is a germ-absorber of "Brownian" particles. In this connection, it is interesting to study the state of this system at large times. It turns out that the limit state is symmetric irrespective of the initial conditions.

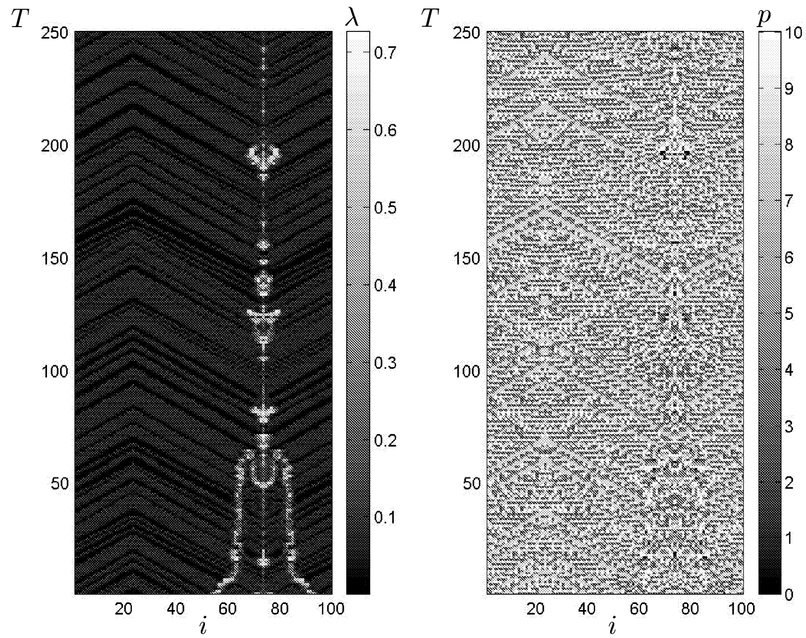


**Fig. 10.** The same as in Fig. 9 in the presence of a defect with $a_2 = 1.99$ in one element ($i = 75$) for $\varepsilon = 0.1$ after $5 \cdot 10^7$ preliminary iterations.

Figure 10 demonstrates a detailed behavior pattern after $5 \cdot 10^7$ iterations, where, for convenience, the defect with $a_2 = 1.97$ is moved to the element with $i = 75$. We see that the chaotic particles are created on the defect in pairs, and they move in trajectories symmetric with respect to the defect. They also disappear in collisions with each other and with the element having the defect. We note that the chaotic elements cause a slight synchronization detuning in all elements. As is seen in Fig. 10, this detuning propagates in space along different directions and converges at the point (on the ring) opposite the defect.

The presented examples show that inhomogeneous map lattices have an absolutely unpredictable variety of possible modes of motion, which can be effectively demonstrated using the figure $\lambda$ introduced in Sec. 2.

## 5. Conclusion

Distributed media can be rather well approximated by space- and time-discrete coupled-map systems or lattices. But knowing the parameters determining the global behavior of coupled-map lattices at asymptotically large times is not very interesting. It is more important to find characteristics determining the local properties of these systems and their evolution in time.

In this paper, a new analysis method is developed for coupled-map chains that permits visualizing the behavior of the individual elements and the dynamics of the entire system as a whole. It is used to investigate the behavior of both homogeneous systems of diffusively coupled one-dimensional quadratic maps and systems with spatial inhomogeneity. Different types of inhomogeneity have been thus considered, periodic inhomogeneity throughout the space in the form of defects in several consecutive elements and a defect in a single element. We showed that the presence of different types of inhomogeneities in the model can substantially change the behavior of the entire system. In particular, for some definite values of the parameters and diffusion coefficients, the dynamics of a inhomogeneous chain with alternating defects are synchronized irrespective of the initial conditions. The peculiarity of this chain is that the homogeneous systems with parameter values determining the inhomogeneities show space–time chaos. We can say that the synchronization of the complex behavior develops by introducing a spatially periodic inhomogeneity.

In conclusion, we note that analyzing the dynamics of coupled-map systems is a very complicated problem and, as a rule, even the consideration of homogeneous discrete models encounters essential difficulties. Nevertheless, the suggested criterion for local analysis permits simplifying the investigation and visually demonstrating the behavior of lattices as a whole.

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