Eur. Phys. J. B **79**, 79–84 (2011)
DOI: 10.1140/epjb/e2010-90983-8

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Exact analytical solutions for nonlinear waves in the inhomogeneous Fisher-Kolmogorov equation

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Received 18 December 2009 / Received in final form 31 October 2010
Published online 10 December 2010 –

Abstract. Exact analytical solutions of the reaction-diffusion equations with spatial inhomogeneous reaction and diffusion coefficients are found. It is shown that the space-oscillating approximate solution of a traveling wave type [P.K. Brazhnik, J.J. Tyson, SIAM J. Appl. Math. **60**, 371 (1999)] is the exact one for the inhomogeneous Fisher-Kolmogorov equation in two dimensions.

1 Inhomogeneous Fisher-Kolmogorov equation

As is known, many physical, chemical, biological and other non-equilibrium phenomena are spatially extended, i.e. they should be described in the framework of the theory of distributed media. As a rule, distributed media consist of a set of elements interacted with each other locally and thus, they are some approximations of natural systems [1–3]. It is well known that these media can be both passive and active ones. In the first case extended excitations applied to an element of the medium are damped. Active distributed systems, in contrast to passive media, possess the property of the existence of sustained waves. This means that the external perturbation may excite the neighboring elements such that the wave propagation is initiated. Such a process is provided by parameters that are responsible for the “relax time” which is said to be the refractory period. In this case we have an ordered spatial spreading of the excitation waves.

In spite of the fact that such a simplification leads to a certain approximation, it can describe main natural processes in the active distributed systems including nonlinear wave dynamics, pattern formation and so on.

According to the behavior of an isolated element which forms the medium, all active media can be classified as bistable, excitable and oscillatory ones. If a medium consists of bistable elements then large enough perturbations can lead to formation and propagation of trigger waves. For a medium of excitable elements the phenomenon of traveling waves is typical. In oscillatory media one can observe time-dependent activity patterns which is said to be phase dynamics. Detailed analysis of the dynamics of wave propagation in active media is presented in references [2,3].

Formation and propagation of waves can be described by means of partial differential equations of a general reaction-diffusion type [1]. One of the well-known models here is the Fisher-Kolmogorov (FK) equation, or the Fisher-Kolmogorov-Petrovskii-Piskunov equation [4,5], which has been proposed by Fisher to describe the spread of an advantageous gene in a population, and Kolmogorov with coworkers who obtained the basic analytical results for this equation. In the two-dimensional case this equation has the following form:

$$\frac{\partial u}{\partial t} = au(1-u) + D\triangle u,$$

where u is the frequency of the mutant gene, a is some intrinsic coefficient, $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator, and D is the matrix of diffusion constants.

In the overwhelming majority cases the authors investigate the homogeneous medium (see, e.g., [6,7] and Refs. cited therein). At the same time, in the real complex systems more adequate physical models are inhomogeneous one, when the medium properties depend on spatial components. Therewith, inhomogeneities may be spatially periodic, random, localized and so on. In all these cases inhomogeneities complicate the spatio-temporal dynamics [8] and lead to the non-trivial behavior of the medium [2,9,10].

For example, Shigesada et al. [11] studied how inhomogeneities influence on the population invasions and the velocity of periodic waves in the inhomogeneous FK equation when the reaction term and the diffusion coefficient

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depend on the spatial coordinate, i.e.

$$\frac{\partial u}{\partial t} = [a(x) - bu]u + \frac{\partial}{\partial x}\left[D(x)\frac{\partial u}{\partial x}\right], \qquad b = \text{const.}, \quad (1)$$

where the inhomogeneity appears in the diffusion coefficient $D(x)$ and the intrinsic growth rate $a(x)$. The system behavior has been analyzed by a dispersion relation. Later (see [12]) this analysis has been generalized for 2D case.

Méndez et al. [13] investigated the wave front dynamics that was described by following equations with smooth inhomogeneities:

$$\frac{\partial u}{\partial t} = U(\varepsilon x)f(u) + \frac{\partial^2 u}{\partial x^2}, \quad (2)$$

or

$$\frac{\partial u}{\partial t} = f(u) + \frac{\partial}{\partial x}\left[D(\varepsilon x)\frac{\partial u}{\partial x}\right],$$

where the function $f(u)$ satisfies $f(0) = f(1) = 0$ and ε is a small parameter. Using the singular perturbation analysis and the Hamilton-Jacoby dynamics the authors obtained the expressions for the front velocity as functions of time.

Also, Keener [14] considered the equations similar to (2) as a model for waves in an excitable chemical medium with discrete release sites and derived analytical expressions for the waveform and for the speed of propagation.

In the present paper we consider several kinds of the FK equation with the smooth inhomogeneous reaction function:

$$\frac{\partial u}{\partial t} = [a(x) - b(x)u]u^n + \frac{\partial}{\partial x}\left[D(x)\frac{\partial u}{\partial x}\right], \qquad n \geq 1. \quad (3)$$

We will focus our attention on periodic functions so that $a(x), 1/b(x) \propto \cos x$ and study particular cases which allows us to obtain exact analytical solutions. In some sense, this paper is some continuation of our investigations devoted to excitable media [9,10].

2 Nonlinear wave solutions

In this section we analytically study the inhomogeneous Fisher-Kolmogorov equation for one- and two-dimensional cases.

2.1 One-dimensional case

First of all let us study a one-dimensional case and consider equations with $a, D = \text{const.}$ and $b = b(x)$. Following Shigesada et al. [11] we suppose that the environment is composed of two kinds of patches in a periodic manner. Then the simplest inhomogeneity can be introduced as a periodic function in parameter space[1]. In such a system the spread of population waves located in a bounded region leads to formation and development of periodic patterns moving with a constant velocity [11]. Because we restrict ourselves by a finite continuous function, it is quite natural to choose the simplest form as follows: $b(x) = b_0/\cos x$ with $b_0 = \text{const.}$

Then the dimensionless equation (3) at $n = 1$ with eliminated constant b_0 can be written as

$$\frac{\partial u}{\partial t} = au - \frac{u^2}{\cos x} + D\frac{\partial^2 u}{\partial x^2}. \quad (4)$$

To solve this equation, let us consider its more general form:

$$\frac{\partial u}{\partial t} = au - \frac{u^2}{K(x)} + D\frac{\partial^2 u}{\partial x^2}. \quad (5)$$

Suppose that the function $u(x,t)$ can be presented as a product of two parts, $X(x)$ and $T(t)$, i.e.

$$u(x,t) = X(x)T(t). \quad (6)$$

In this case the substitution of (6) into (5) gives:

$$XT_t = aXT - \frac{X^2T^2}{K} + DX_{xx}T, \quad (7)$$

where subscripts on X and T denote the corresponding derivatives.

Since the functions $X(x)$ and $K(x)$ are arbitrary ones, we may consider a particular case when they are equal to each other: $X(x) = K(x)$. Then both variables in the relation (7) can be easily separated so that we get two uncoupled equations for $X(x)$ and $T(t)$, respectively:

$$DX_{xx} = (q-a)X,$$
$$T_t = qT - T^2, \quad (8)$$

where q is an unknown separation constant.

One can see that general solutions of (8) are the following:

$$X(x) = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x}, \quad \lambda_{1,2} = \pm\sqrt{\frac{q-a}{D}}, \quad q > a,$$

$$X(x) = \bar{A}_1\cos\bar{\lambda}x + \bar{A}_2\sin\bar{\lambda}x, \quad \bar{\lambda} = \sqrt{\frac{a-q}{D}}, \quad q < a,$$

$$X(x) = \bar{A}x + A_0, \quad q = a, \quad (9)$$

and

$$T(t) = \frac{q}{1 + e^{c_0 - qt}}, \quad q \neq 0,$$

$$T(t) = \frac{1}{c_0 - t}, \quad q = 0, \quad (10)$$

where the constant c_0 is defined by initial conditions and A are the integration constants.

Let us turn now to such a particular solution of (9) when during the spreading of population waves the periodic patterns appear. This situation was described in [11]

[1] In this case the medium may be described as a periodic lattice [9,10].

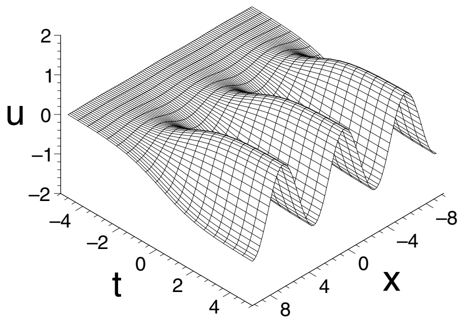

Fig. 1. Propagation of the wave $u = u(x,t)$ described by equation (11) at the fixed parameters $c_0 = 0$ and $a - D = 1$.

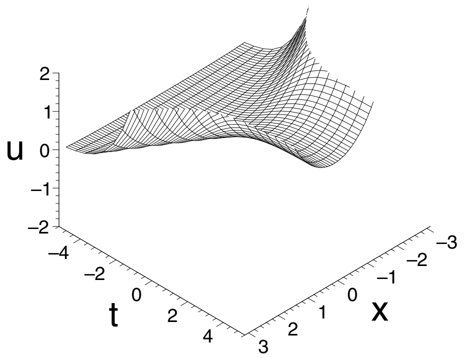

Fig. 2. Propagation of the wave $u = u(x,t)$ described by equation (12) at the fixed parameters $c_0 = 0$ and $a + D = 1$.

on the example of a similar inhomogeneous system. The simplest periodic function is $K(x) = \cos x$. Hence, choosing $q = a - D$ we may write the solution of equation (4) as

$$u(x,t) = \frac{(a-D)\cos x}{1+\exp\left[c_0-(a-D)t\right]}. \qquad (11)$$

It can be checked by a direct substitution. The obtained relation is a particular solution of a certain specific case. It is necessary to emphasize that it is an *exact analytical solution*. This solution describes the nonlinear waves which are periodic in space, and their temporal dynamics is similar to the kink wave. It should be noted that the time dependence in the relation (11) shows resemblance with spatial dependence found in [15].

An example of the solution (11) is shown in Figure 1. One can see that the form of this solution is a periodic wave with spatial oscillations arising from uniform stationary state at $u = 0$.

If inhomogeneity has the type of $K(x) = \cosh x$ (see the first expression in the relation (9)), the solution takes the following form:

$$u(x,t) = \frac{(a+D)\cosh x}{1+\exp\left[c_0-(a+D)t\right]}, \qquad (12)$$

where $q = a + D$ is chosen. It looks like an U-formed pattern (Fig. 2), i.e. the wave consists of two wings connecting the asymptotic (homogeneous) state at $t \to -\infty$ and the inhomogeneous state at $t \to +\infty$.

The variable separation procedure which we used to obtain the exact solution of the equation with the quadratic nonlinearity can be applied to solve equations with the other nonlinearity power laws, i.e.

$$\frac{\partial u}{\partial t} = au - \frac{u^n}{K^{n-1}(x)} + D\frac{\partial^2 u}{\partial x^2}, \quad n > 1.$$

For example, in the case of cubic nonlinearity of the reaction function and $K^2(x) = 2\cos^2(x/2)$, where the constant b is omitted, the corresponding equation should have the form:

$$\frac{\partial u}{\partial t} = au - \frac{u^3}{1+\cos x} + D\frac{\partial^2 u}{\partial x^2}.$$

Then the variable separation method gives the following exact solution:

$$u(x,t) = \frac{\sqrt{a-D/4}\,\cos(x/2)}{\sqrt{1/2+\exp\left[c_0-2(a-D/4)t\right]}}.$$

However, in fact, this solution looks like the expression (11).

All the solutions obtained above admit the negative values of the function u. However, in the context of the population dynamics this function should be non-negative, i.e. in the simplest case it should have the form $u(x,t) \propto 1+\cos x$ instead of $\cos x$. Therefore, to find the exact solution for such "real" waves it is necessary to consider more general variant of the FK equation with inhomogeneous reaction and diffusion terms. In other words, we should analyze the equation

$$\frac{\partial u}{\partial t} = [a(x) - b(x)u]u + \frac{\partial}{\partial x}\left[D(x)\frac{\partial u}{\partial x}\right] \qquad (13)$$

and suppose (for the simplicity only) that all inhomogeneities have similar non-negative dependencies:

$$a(x) = a_0(1+\cos x),$$
$$b(x) = b_0/(1+\cos x),$$
$$D(x) = D_0(1+\cos x), \qquad (14)$$

where a_0, b_0 and D_0 are some constants. In this case we should try to find the solution as follows:

$$u(x,t) = (1+\cos x)T(t).$$

Now, substituting this relation into (13), taking into account (14) and collecting similar terms we get $a_0 = 2D_0$. Then, eliminating the factor $1+\cos x$ we obtain the equation for T only: $T_t = 3D_0T - b_0T^2$. So, we find its exact solution:

$$T(t) = \frac{3D_0}{b_0+\exp\left(c_0-3D_0t\right)}.$$

Hence, we arrive at the final relation:

$$u(x,t) = \frac{3D_0(1+\cos x)}{b_0+\exp\left(c_0-3D_0t\right)}.$$

One can be easily seen that this obtained exact solution is, in fact, the expression (11) which we derived above, but shifted to the area of positive values.

2.2 Two-dimensional case

Now first let us consider the simplest two-dimensional case when the inhomogeneity can be factorized such that $K(x, y) = M(x)N(y)$. Then the FK equation is written as follows:

$$\frac{\partial u}{\partial t} = au - \frac{u^2}{M(x)N(y)} + D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

If we express the solution as $u(x, y, t) = X(x)Y(y)T(t)$, then we can rewrite this equation in the form

$$XYT_t = aXYT - \frac{X^2Y^2T^2}{MN} + D\left(X_{xx}YT + XY_{yy}T\right)$$

and separate the variables X, Y and T as before. For $M(x) = \cos x$ and $N(y) = \cos y$ this approach gives the following result:

$$u(x, y, t) = \frac{(a - D)\cos x \cos y}{1 + \exp\left[c_0 - (a - D)t\right]}$$

which is a trivial generalization of the solution (11) for two-dimensional case.

When the inhomogeneity in the reaction term depends on the variable x, the corresponding equation has the form:

$$\frac{\partial u}{\partial t} = au - \frac{u^2}{\cos x} + D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

Then, excluding from the function $u(x, y, t)$ its dependence on x and using the substitution

$$u(x, y, t) = v(y, t)\cos x,$$

we get the homogeneous FK equation

$$\frac{\partial v}{\partial t} = (a - D)v - v^2 + D\frac{\partial^2 v}{\partial y^2}$$

which has the well-known analytical solution for the wave of the unchanged profile moving with the constant velocity [16]:

$$v(y, t) = \frac{a - D}{\left[1 + \exp\left(\sqrt{\frac{a - D}{6}}\xi\right)\right]^2}.$$

Here $\xi = y/\sqrt{D} - ct$ is the coordinate of the wave in the traveling frame (traveling wave coordinate) and c is the wave velocity. This exact solution is a specific case of the traveling front which has the velocity $c = 5\sqrt{(a - D)/6}$ [16].

From this result we finally find that the two-dimensional wave solution is written as

$$u(x, y, t) = \frac{(a - D)\cos x}{\left[1 + \exp\left(\sqrt{\frac{a - D}{6}}\xi\right)\right]^2}. \quad (15)$$

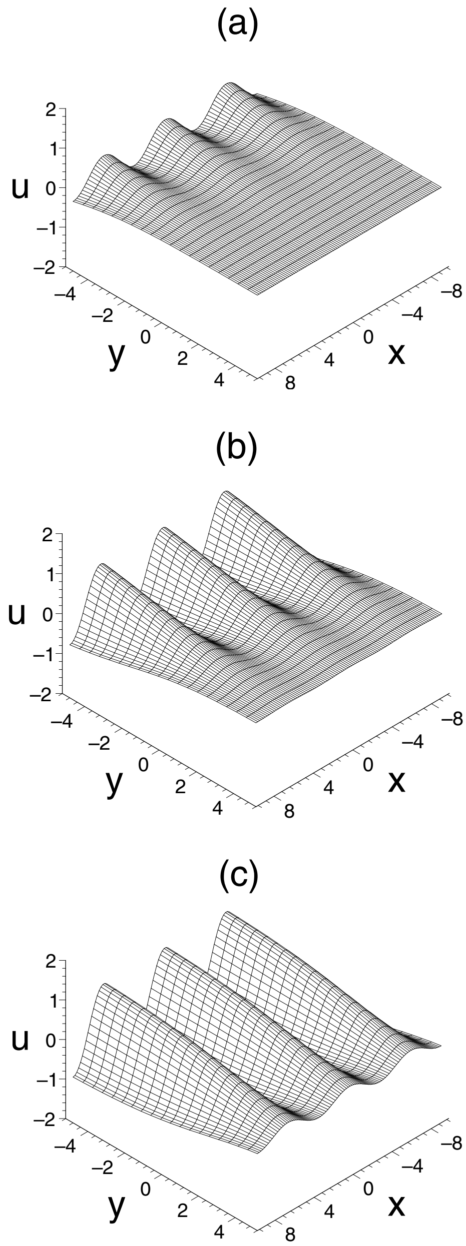

Fig. 3. Propagation of the waves $u = u(x, y, t)$ described by the relation (15) at the fixed values $a - D = 1, D = 1$ and the velocity $c = 5\sqrt{(a - D)/6}$. The waves are shown at the different values of time moments: (a) $t = -2$, (b) $t = 0$ and (c) $t = 2$.

This exact solution is shown in Figure 3. It is obvious that it looks like a spatio-temporal pattern obtained above for the one-dimensional equation (see Fig. 1). However, this solution differs qualitatively from 1D case by the fact that oscillations occur in the (x, y)-plane.

It should be noted that our relation (15) coincides with the spatially oscillating wave solution of the Fisher-Kolmogorov equation obtained by Brazhnik and Tyson (see formulae (3.25) in the paper [17]), if the variables x and y change places. However, we should emphasize that Brazhnik and Tyson have found the *approximate solution*.

It has been derived via the heuristic substitution of the quadratic function by the linear one. In our case the obtained expression is the *exact solution* of the inhomogeneous Fisher-Kolmogorov equation. This exact result we found due to the specific inhomogeneity type which leads to the linear equation for the spatial component[2] $X(x)$, see, e.g. the first equation in the relations (8).

Thus, the inhomogeneity as if "compensates" (or, in other words, linearizes) the effect from the function nonlinearity in the part of its spatial component.

3 Stability of the solutions

In this section we consider questions related to the stability of the obtained exact analytical solutions (11), (12) and (15).

From a general theory of evolution equations of mathematical physics (see [18]), it follows that attractors in such systems are a finite-dimensional (in the sense of Hausdorf) and compact set. Moreover, for equations of the reaction-diffusion type it was proven that with time the initial infinite-dimensional dynamics tends to finite-dimensional behavior on the corresponding attractor.

To be more precise, consider an evolution system in a general form $\partial_t u = \Lambda u$, $u|_{t=0} = u_0$. For this system there is a semi-group of operators $\{S_t\}$, $t \geq 0$, such that for every initial condition u_0 it puts into correspondence the solution $u(t)$, $S_t u_0 = u(t)$ at any t. At $t \to \infty$ this solution $u(t)$ will tend to some set $\mathcal{A}$. If this set is a compact, and it is invariant with respect to $\{S_t\}$, i.e. $S_t\mathcal{A} = \mathcal{A}$, then $\mathcal{A}$ is said to be attractor.

As follows from a rigurous theory [18], generic systems of the reaction-diffusion type (including equations considered in the present paper), have a unique solution. Moreover, for $t > 0$ the corresponding semi-group $\{S_t\}$ is bounded for which there exists an absorbing compact subset. Using this result one can prove [18] that $\{S_t\}$ possesses an attractor $\mathcal{A}$.

As a rule, in applications to real systems of mathematical physics the notion of a maximal (or global) attractor is used. Attractor $\mathcal{A}$ for which there exists an arbitrary bounded subset W such that under the action of $\{S_t\}$ the distance $\mathrm{dist}(S_t W, \mathcal{A})$ tends to zero at $t \to \infty$ is called maximal attractor.

At some additional conditions of the system differentiability the described attractors consist of a union of smooth finite-dimensional manifolds, i.e. they are regular sets.

In our case the analytically constructed solutions $u(x,t)$ for one- and two-dimensional media have the forms (11), (12) and (15). As follows from the presented argumentation, they are attractors and therefore cannot be destroyed by small enough perturbations or the presence of external noises of a low level.

[2] For the temporal component $T(t)$ the equation can have a nonlinear form (see the second equation in (8)).

4 Extended Fisher-Kolmogorov equation with inhomogeneity

In the present paper we studied some partial cases of the inhomogeneous Fisher-Kolmogorov equation and found their *exact solutions* which are spatially oscillatory structures and U-formed patterns. These wave solutions do not depend on the initial conditions, i.e. the propagating waves have the fixed shape (profile). The characteristic feature of the obtained solution consists of the fact that at the fixed diffusion coefficient and the reaction constant the wave velocity has a certain value for each particular solution. Also it should be emphasized that these are the exact expressions, i.e. for them there are no some restrictions for the model parameters.

The obtained exact solutions can be chosen as a starting point for the investigations of more complicated systems. For example, by the same way we can derive a solution of the inhomogeneous extended FK equation [19,20] with quadratic (or cubic) nonlinearity

$$\frac{\partial u}{\partial t} = au - \frac{u^2}{\cos x} + D\frac{\partial^2 u}{\partial x^2} - \gamma\frac{\partial^4 u}{\partial x^4}, \quad \gamma = \text{const.}$$

Really, in this case the expression (11) remains the same only with the substitution $D \to D + \gamma$.

However, the considered procedure which is reduced to the compensation of a certain part of nonlinear function by some inhomogeneity may be applied to the restricted class of equations. In other cases, to find the corresponding solutions of nonlinear equations alternative methods can be used. Among them Cole-Hopf transformations [21,22] or piecewise linear approximations of the nonlinear reaction term [23–25] are often utilized. Both of these approaches allows us to construct the corresponding solutions for equations and systems of equations (see, e.g. [21,22]). However, the given ways also have some restrictions. In particular, by means of a piecewise linear approximation it is possible to reduce the considered problem to the solution of a polynomial equation for eigenvalues, that is not so simple. That is the reason why the obtained in the present research solutions for the reaction-diffusion equations are very important results from theoretical and applied points of view.

Some parts of the present study were made with V.M. Kenkre from University of New Mexico, USA. One of the authors (E.P.Z.) thanks him and P.K. Brazhnik for useful comments.

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