MODEL OF A SPATIALLY INHOMOGENEOUS ONE-DIMENSIONAL ACTIVE MEDIUM

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We investigate the dynamics of one-dimensional discrete models of a one-component active medium analytically. The models represent spatially inhomogeneous diffusively concatenated systems of one-dimensional piecewise-continuous maps. The discontinuities (the defects) are interpreted as the differences in the parameters of the maps constituting the model. Two classes of defects are considered: spatially periodic defects and localized defects. The area of regular dynamics in the space of the parameters is estimated analytically. For the model with a periodic inhomogeneity, an exact analytic partition into domains with regular and with chaotic types of behavior is found. Numerical results are obtained for the model with a single defect. The possibility of the occurrence of each behavior type for the system as a whole is investigated.

1. Introduction

The approximate representation of continuous media by their discrete analogues is a rather effective research method. The conversion to the discrete form can be either purely spatial or both spatial and temporal. In the spatial discretization, the initial system is approximated by a finite or countable set of elements with a certain form of coupling between them. Every element represents a dynamic system with a small number of variables. If, in addition, the dynamic system is determined by a map, i.e., by a transformation with discrete time, then the discretization is called spatial–temporal. The spatial–temporal discrete models are called *net* or *lattice* models. Many problems in the nonlinear theory of a continuous medium can be reduced to problems with discrete models. For example, some problems in statistical physics can be effectively solved through the representation of the medium by its discrete space–time approximation [1-4], excitable continuous systems are often described in terms of their discrete analogues [4, 5], etc. (cf. [6–13] and the references therein). Moreover, any sort of numerical analysis of continuous systems is always connected with discrete space–time systems because any numerical procedure is based on a finite difference scheme.

There is a variety of space–time lattice forms. The most common are the lattices where every element somehow interacts only with its nearest neighbors (see, e.g., [8–13] and the references therein). Another form of coupling is global interaction in which every pair of elements is connected [14-16]. In the case of a local interaction, diffusion coupling between the elements is normally considered [1, 2, 4, 8–13] (see also the references therein). This form of coupling is mainly used to model the phenomena related to space–time chaos, structure generation, and self-organization.

If the value of a governing parameter of the family of maps constituting a lattice is constant for the entire lattice, the lattice is called *homogeneous*. Varying this parameter and the value of the diffusion (or the coupling constant), we can investigate the phase diagram, the possible phase transitions, and the related space–time patterns. If the parameters of the maps are different, then the net is inhomogeneous. Such nets are much more difficult to study. Almost all previous works dedicated to diffusively interacting maps

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were restricted to a study (usually a numerical study) of homogeneous nets. From the physical standpoint, however, it is obvious that the space homogeneity (or, in this case, the identity of all elements) is a strong idealization adopted to simplify the analysis. It is therefore interesting to investigate the qualitative changes in the system dynamics related to the inhomogeneity. The latter can be of different types ranging from individual defects to a periodic inhomogeneity of the entire space.

The present work is focused on spatially inhomogeneous one-dimensional systems (i.e., chains) of diffusively connected piecewise-linear one-dimensional maps. The term map here refers to a transformation of an interval I into itself: $T_a \colon I \to I$, where $I = [\alpha, \beta] \in \mathbb{R}^1$, $x \mapsto G(x, a)$, $x \in I$, $G(x, a)$ is a function, and a is a parameter (or, generally, a set of parameters). In terms of the iterations k, this can be written as $x(k+1) = G(x(k), a)$, $x \in I$. The inhomogeneities are given as maps with different parameters. The maps themselves (i.e., the functions G) are chosen such that the systems represent models of a one-component active medium [17]. Based on an exact calculation of the Lyapunov indices, we investigate different regimes in the behavior of the system with a periodic inhomogeneity and of the system with a single defect. The phase space structure of these systems is also described.

2. Models of spatially inhomogeneous active media

The main result in this section is the description of the dynamics of two types of inhomogeneous active media. We consider a model representing a one-dimensional annular lattice of concatenated maps with a periodic inhomogeneity and a chain with a single defect.

2.1. Homogeneous medium.
We examine a system of N diffusively concatenated maps of the form

$$x_n(k + 1) = G\big(x_n(k), \alpha, \gamma\big) + \varepsilon\big(x_{n+1}(k) - 2x_n(k) + x_{n-1}(k)\big) \tag{1}$$

with the periodic boundary conditions $x_{n+N}(k) \equiv x_n(k)$, where $n = 1, 2, \ldots, N$ is the discrete space coordinate, $k = 0, 1, 2, \ldots$ is the discrete time coordinate, $\varepsilon > 0$ is the diffusion coefficient (which we assume to be constant), and $\alpha > 0$ and γ are parameters. For the function G, we set

$$G(x, \alpha, \gamma) = x + \alpha F(x) - \gamma = \begin{cases} (1 - 2\alpha)x - \gamma, & x < 1/2, \\ (1 - 2\alpha)x - \gamma + 2\alpha, & x > 1/2. \end{cases} \tag{2}$$

This form of the function $G(x, \alpha, \gamma)$ was chosen because system of maps (1) with elements of form (2) is a discrete realization of the basic model of a one-component one-dimensional active medium described by an equation of the Kolmogorov–Petrovskii–Piskunov form:

$$\frac{\partial U(x, t)}{\partial t} = D \frac{\partial^2 U(x, t)}{\partial x^2} + \Phi(U).$$

As is known, this equation is commonly met in problems in biophysics, combustion theory, chemical kinetics (e.g., the description of the Belousov–Zhabotinskii reaction), solid state electronics, etc. (see, e.g., [18–21]).

It is easy to see that $G(x, \alpha, \gamma)$ is a piecewise-linear function whose graph has a constant slope equal to $(1 - 2\alpha)$. Depending on the parameter α, the map generated by the function G, i.e., $x \mapsto G(x, \alpha, \gamma)$, can manifest qualitatively different types of behavior. For $0 < \alpha < 1$ (Fig. 1a), its behavior is regular, and depending on γ, the map can have one or two stable points (the points A and B) attracting almost all points of the phase space. For $\alpha > 1$, the absolute value of the slope of the graph of the map exceeds unity (see Fig. 1b), which corresponds to an exponential divergence of neighboring trajectories. In this case, if the motion of a phase point is finite, the dynamics is chaotic.

Homogeneous model (1) composed of concatenated functions of form (2) was described in detail in [17]. We recall the principal results of this study. System (1), (2) can be interpreted as an N-dimensional
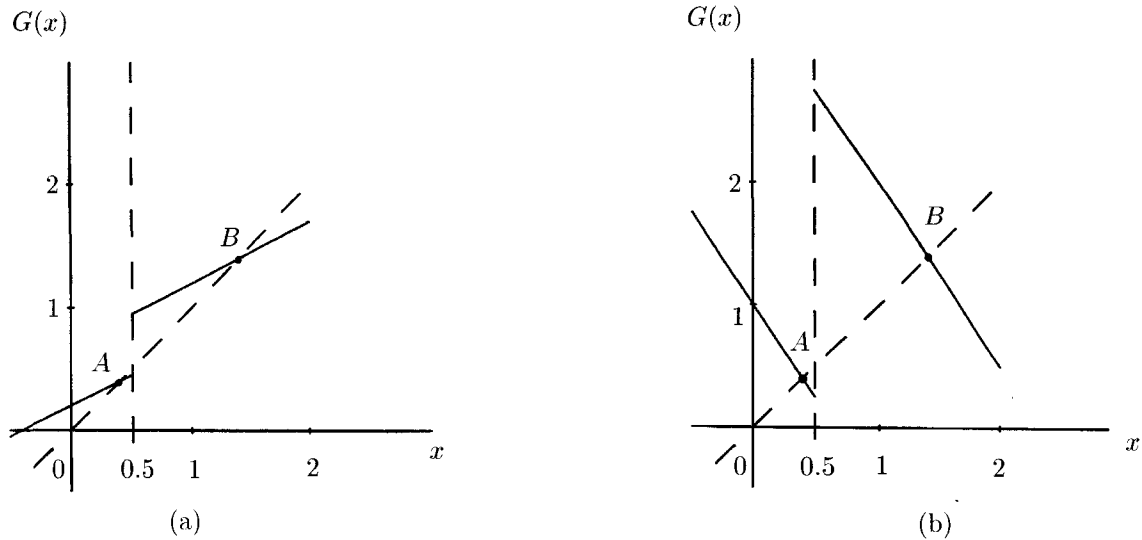
Fig. 1. The map $x \mapsto G(x, \alpha, \gamma)$: (a) for $\alpha = 0.25$ and $\gamma = -0.2$; (b) for $\alpha = 125$ and $\gamma = -1$.

piecewise-linear map $f: \mathbb{R}^N \to \mathbb{R}^N$. Because $G(x, \alpha, \gamma)$ is a piecewise-linear function with a constant derivative, the differential D_f of the map f is a matrix with constant coefficients. It is therefore easy to find the Lyapunov indices λ_i of this map. If ρ_s are the eigenvalues of D_f, then $\lambda_s = \log|\rho_s|$. If any eigenvalue ρ_s lies outside the unit circle on the complex plane, then all trajectories of the map f are unstable. Otherwise, the model dynamics is regular. The problem of the system behavior is therefore connected with investigating the location of the roots ρ_s. To calculate ρ_s, we express the characteristic polynomial for the N-dimensional map differential in terms of determinants of three-diagonal matrices with different dimensions. We can obtain recursive relations for these matrices. Interpreting these relations as finite-difference equations with given initial conditions, it is possible to find the solutions, which are proportional to the Chebyshev polynomials of a linear function of the eigenvalues of D_f. It can be shown that because of the properties of Chebyshev polynomials, the characteristic polynomial reduces to a quadratic third-order polynomial of the known function ρ_s. Calculating the values of ρ_s and analyzing their locations on the complex plane lead to a partition of the parameter space of map system (1), (2) into two domains (Fig. 2):

1. Domain D_1 is determined by parameters α and ε such that the absolute values of all eigenvalues of the map f differential are smaller than unity, and the model dynamics is therefore regular.
2. Domain D_2 is determined through the condition that the set of eigenvalues ρ_s contains at least one root satisfying the inequality $|\rho_s| > 1$. In this case, generally, the dynamics can be infinite. However, if the motion is finite, the behavior of system (1), (2) is said to be *chaotic* [9–11].

System (1), (2) is spatially homogeneous in the sense that all its elements (i.e., the maps $x \mapsto G(x, \alpha, \gamma)$) are equal. However, our goal is to investigate the spatially inhomogeneous diffusive model. The inhomogeneity can be related either to different map types acting in different space points or (if the maps for all the elements are equal) to different map parameters.

In this analytic study, we restrict ourselves to the latter case, i.e., we assume that every element is a piecewise-linear map $x_i \mapsto G(x_i, \alpha_i, \gamma)$ (see Eq. (2)) but, in contrast to the homogeneous case, there are two element types distinguished by the values of the parameters α_i of the function $G(x_i, \alpha_i, \gamma)$.[2] We let the parameter α correspond to one type of the elements and the parameter β correspond to the other type.

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[2]Generally, the elements can also differ by the value of γ. However, it is shown below that the dynamics of both the homogeneous and the inhomogeneous systems does not depend on γ.
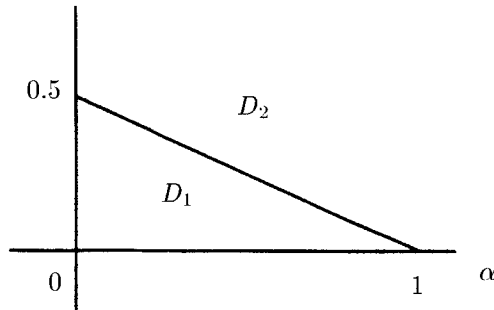
Fig. 2. Parameter space of a homogeneous annular chain of diffusively concatenated maps (1), (2).

As in homogeneous case (1), we take the chain in the form of a ring, i.e., $x_{n+N}(k) \equiv x_n(k)$.

We investigate two qualitatively different cases in detail: the model with spatially periodic defects and the model with a single defect.

2.2. Annular model with a spatially periodic inhomogeneity. The term spatially periodic inhomogeneity refers to the case where the values of the parameter of the function G change from one element to another periodically, i.e., the parameters vary as $\alpha, \beta, \alpha, \beta, \ldots$. The dynamics of a chain with such an inhomogeneity can be fully described analytically.

We consider a concatenated one-dimensional net of the form

$$x_n(k+1) = \begin{cases} G\big(x_n(k), \alpha, \gamma\big) + \varepsilon\big(x_{n+1}(k) - 2x_n(k) + x_{n-1}(k)\big) & \text{for odd } n, \\ G\big(x_n(k), \beta, \gamma\big) + \varepsilon\big(x_{n+1}(k) - 2x_n(k) + x_{n-1}(k)\big) & \text{for even } n, \end{cases} \tag{3}$$

where $x_{n+N}(k) \equiv x_n(k)$, the function G is still given by relation (2), and N is even (otherwise chain (3) cannot be converted to the form of a ring). We seek the values of the parameters α, β, and ε corresponding to qualitatively different regimes of behavior of periodically inhomogeneous annular model (3), (2). We calculate the Lyapunov indices of system (3), (2) using the technique developed in [17].

Taking the boundary conditions into account, we express the differential of the N-dimensional map generated by a system of form (3) as

$$\mathrm{D}_f \equiv Q_N = \begin{pmatrix} 1 - 2\varepsilon - 2\alpha & \varepsilon & 0 & \cdots & \varepsilon \\ \varepsilon & 1 - 2\varepsilon - 2\beta & \varepsilon & \cdots & 0 \\ 0 & \varepsilon & 1 - 2\varepsilon - 2\alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & 0 & 0 & \cdots & 1 - 2\varepsilon - 2\beta \end{pmatrix}.$$

To find the eigenvalues ρ_s, $s = 1, 2, \ldots, N$, of the matrix Q_N, we calculate the determinant

$$\det(Q_N - \rho I_N) = \begin{vmatrix} 2z_1\varepsilon & \varepsilon & 0 & 0 & \cdots & \varepsilon \\ \varepsilon & 2z_2\varepsilon & \varepsilon & 0 & \cdots & 0 \\ 0 & \varepsilon & 2z_1\varepsilon & \varepsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \varepsilon \\ \varepsilon & 0 & 0 & 0 & \cdots & 2z_2\varepsilon \end{vmatrix},$$

where z_1 and z_2 are determined from the relations $1 - 2\varepsilon - 2\alpha - \rho = 2z_1\varepsilon$ and $1 - 2\varepsilon - 2\beta - \rho = 2z_2\varepsilon$. Expanding the determinant with respect to the first row, we obtain

$$\det(Q_N - \rho I_N) = 2z_1\varepsilon B_{N-1} - \varepsilon^2 B_{N-2} - 2\varepsilon^N - \varepsilon^2 \widehat{B}_{N-2},$$

where

$$B_N = \begin{vmatrix} 2z_1\varepsilon & \varepsilon & 0 & \cdots & 0 \\ \varepsilon & 2z_2\varepsilon & \varepsilon & \vdots & 0 \\ 0 & \varepsilon & 2z_1\varepsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2z_2\varepsilon \end{vmatrix}, \qquad \widehat{B}_N = \begin{vmatrix} 2z_2\varepsilon & \varepsilon & 0 & \cdots & 0 \\ \varepsilon & 2z_1\varepsilon & \varepsilon & \cdots & 0 \\ 0 & \varepsilon & 2z_2\varepsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2z_1\varepsilon \end{vmatrix}.$$

It is easy to see that $\widehat{B}_N(z_1, z_2)$ can be obtained from $B_N(z_1, z_2)$ through the substitution $z_1 \leftrightarrow z_2$, and we therefore use a recursive relation to find $B_N(z_1, z_2)$. Expanding B_N with respect to the first row, we obtain

$$B_N = \begin{cases} 2z_1\varepsilon B_{N-1} - \varepsilon^2 B_{N-2} & \text{for even } N, \\ 2z_2\varepsilon B_{N-1} - \varepsilon^2 B_{N-2} & \text{for odd } N. \end{cases}$$

It is now easy to express the odd-order determinants through the even-order determinants:

$$B_{2N+1} = \frac{B_{2N+2} + \varepsilon^2 B_{2N}}{2z_1\varepsilon}. \tag{4}$$

We then obtain a recursive relation for the even-order determinants:

$$B_{2N} = \varepsilon^2(4z_1 z_2 - 2)B_{2N-2} - \varepsilon^4 B_{2N-4}. \tag{5}$$

Relation (5) can be interpreted as a difference equation with the initial conditions

$$B_0 = 1, \qquad B_2 = \varepsilon^2(4z_1 z_2 - 1). \tag{6}$$

Further, solving system (5), (6) and taking Eq. (4) into account, we obtain

$$B_N(z_1, z_2) = \begin{cases} \varepsilon^N U_N(z) & \text{for even } N, \\ \varepsilon^N \sqrt{\dfrac{z_2}{z_1}} U_N(z) & \text{for odd } N, \end{cases}$$

where $U_N(z)$ is a Chebyshev polynomial of the second kind. In turn, the relations for $\widehat{B}_N(z_1, z_2)$ have the form

$$\widehat{B}_N(z_1, z_2) = \begin{cases} \varepsilon^N U_N(z) & \text{for even } N, \\ \varepsilon^N \sqrt{\dfrac{z_1}{z_2}} U_N(z) & \text{for odd } N. \end{cases}$$

Using the relations for B_{N-1}, B_{N-2}, and $\widehat{B}_{N-2}$, we find

$$\det(Q_N - \rho I_N) = 2\varepsilon^N \left(z U_{N-1}(z) - U_{N-2}(z) - 1 \right). \tag{7}$$

It is now easy to find the eigenvalues of the differential of map (3), (2). It follows from Eq. (7) that

$$zU_{N-1}(z) - U_{N-2}(z) - 1 = 0 \qquad (8)$$

or $T_N(z) - 1 = 0$, where we use the relation between the second-kind Chebyshev polynomials $U_N(z)$ and the first-kind Chebyshev polynomials $T_N(z)$. Because

$$T_N(z) = \frac{\left(z - \sqrt{z^2 - 1}\right)^N + \left(z - \sqrt{z^2 - 1}\right)^{-N}}{2}$$

(where $z^2 = z_1 z_2$), if we set $t = z - \sqrt{z^2 - 1}$, we obtain

$$\frac{1}{2}\left(t^N + \frac{1}{t^N}\right) - 1 = 0$$

or $(t^N - 1)^2 = 0$ from Eq. (8). Consequently, $t_s = e^{i2\pi s/N}$ and $z_s = \cos(2\pi s/N)$, $s = 1, 2, \ldots, N$. Recalling that $2z_1\varepsilon \cdot 2z_2\varepsilon = 4z^2\varepsilon^2$, we obtain the simple equation

$$(1 - 2\varepsilon - 2\alpha - \rho)(1 - 2\varepsilon - 2\beta - \rho) = 4\varepsilon^2 \cos^2\left(\frac{2\pi}{N}s\right).$$

Solving this equation with respect to ρ, we find the set of eigenvalues of the differential of map (3), (2):

$$\rho_{1,2}^s = 1 - 2\varepsilon - \alpha - \beta \pm \sqrt{(\alpha - \beta)^2 + 4\varepsilon^2 \cos^2\left(\frac{2\pi}{N}s\right)}, \qquad (9)$$

where $s = 1, 2, \ldots, N/2$. The dynamics of system (3), (2) is fully regular if all values of ρ_s lie inside the unit circle on the complex plane. All $\rho_{1,2}^s$ are real; therefore, the regular-dynamics condition is satisfied if

$$2\varepsilon + \alpha + \beta > \sqrt{4\varepsilon^2 + (\alpha - \beta)^2},$$

$$\varepsilon + \frac{\alpha + \beta}{2} + \sqrt{\varepsilon^2 + \left(\frac{\alpha - \beta}{2}\right)^2} < 1.$$

Because $\varepsilon, \alpha, \beta > 0$, the first inequality of this system is always true. In turn, the condition $\varepsilon + (\alpha + \beta)/2 + \sqrt{\varepsilon^2 + (\alpha - \beta)^2/4} > 1$ is satisfied in the parameter space domain where $\lambda_s > 0$.

The boundary between the domains of qualitatively different (regular or chaotic) dynamics is a surface in the three-dimensional space of the parameters $(\alpha, \beta, \varepsilon)$ given by the expression

$$\varepsilon + \frac{\alpha + \beta}{2} + \sqrt{\varepsilon^2 + \frac{(\alpha - \beta)^2}{4}} = 1. \qquad (10)$$

As an illustration, we consider the sections of the phase space of system (3), (2) by planes $\delta = \text{const}$, where $\delta = \beta - \alpha$ is the inhomogeneity parameter (Fig. 3). Let D_1 be the regular-dynamics domain, and let D_2 be the domain where $\lambda_s > 0$. The boundary between these two domains is determined by an equation that follows from Eq. (10). For comparison, the boundaries separating the domains D_1 and D_2 for the homogeneous ($\delta = 0$) system of concatenated maps (see Fig. 2) are shown in Fig. 3a,b (dashed lines). As follows from Fig. 2, the model with $\delta > 1$ exhibits a chaotic dynamics for any values of ε and α. For $\delta < 1$, there exists a domain D_1 where the dynamics is regular. In a certain range of values of the parameters $0 < \delta < 1$, this domain is a subarea of the regular behavior domain of the homogeneous model. For
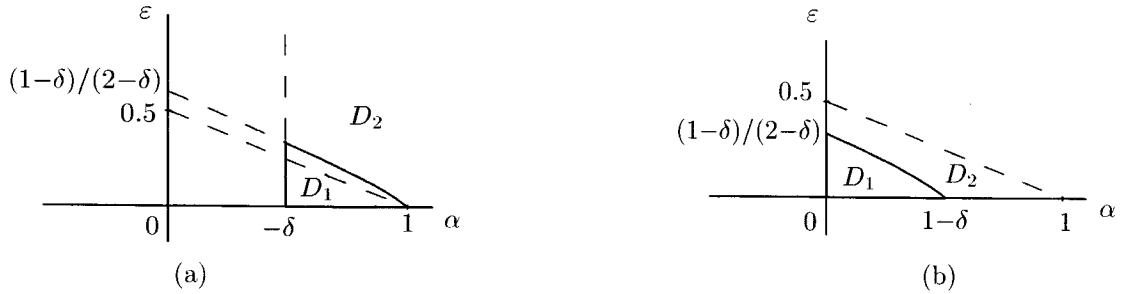
Fig. 3. Sections of the parameter space for model (2), (3) by the planes $\delta = $ const: (a) for $-1 < \delta < 0$; (b) for $0 < \delta < 1$.

$-1 < \delta < 0$, there exist values of the parameters ε and α such that the dynamics of the inhomogeneous model is regular, while the dynamics of the homogeneous one is chaotic. The restriction $\alpha > -\delta$ corresponds to the condition $\beta > 0$. For $\delta < -1$, the domain D_1 vanishes, and model (3), (2) exhibits only chaotic properties for all possible values of ε and α. As directly follows from Fig. 3, there are two qualitatively different possibilities: either the parameter α belongs to the regularity domain, the parameter β belongs to the chaos domain with $\lambda_s > 0$, and the general dynamics is regular or the dynamics of the periodically inhomogeneous model can be chaotic. The realization of one of these two possibilities depends on the value of the diffusion parameter ε.

2.3. Annular model with a single defect. We now consider another interesting spatially inhomogeneous model of diffusively concatenated maps, the system with a single defect. In this case, the parameter β (see formula (2)) corresponds to one of the N elements of the system, while the parameter α corresponds to the remaining $N-1$ elements. Without loss of generality, we can set this single different element to correspond to $n = 1$. Therefore, we examine a model of the form

$$x_n(k+1) = \begin{cases} G\big(x_n(k), \alpha, \gamma\big) + \varepsilon\big(x_{n-1}(k) - 2x_n(k) + x_{n+1}(k)\big), & n = 2, 3, \ldots, N, \\ G\big(x_1(k), \beta, \gamma\big) + \varepsilon\big(x_N(k) - 2x_1(k) + x_2(k)\big), & n = 1, \end{cases} \tag{11}$$

where $x_{n+N} \equiv x_n(k)$. As above, let the function $G(x, \alpha, \gamma)$ be given by expression (2). We determine the differential of the corresponding map for system (11), (2):

$$\mathbf{D}_f \equiv Q_N = \begin{pmatrix} 1 - 2\varepsilon - 2\beta & \varepsilon & 0 & \cdots & \varepsilon \\ \varepsilon & 1 - 2\varepsilon - 2\alpha & \varepsilon & \cdots & 0 \\ 0 & \varepsilon & 1 - 2\varepsilon - 2\alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & 0 & 0 & \cdots & 1 - 2\varepsilon - 2\alpha \end{pmatrix}. \tag{12}$$

If the technique used above to find the Lyapunov indices (see Secs. 2.1 and 2.2 and also [17]) is applied to model (11), (2), then the characteristic equation for map differential (12) leads to the necessity to find the roots of a polynomial of order $2N+2$. This can only be done numerically. We therefore use a somewhat different approach for this model.

We estimate the regular-dynamics domain of system (11), (2) using the estimate for the eigenvalues of matrix (12) based on the Guershgorin theorem [22]. According to this theorem, all eigenvalues of a matrix $A = \{a_{ij}\}_{n \times n}$ belong to a union of circles on the complex plane:

$$\rho_s \in \bigcup_{i=1}^{n} \{z \in \mathbb{C} : \|z - a_{ij}\| \leq R_i'\}, \quad s = 1, 2, \ldots, N,$$
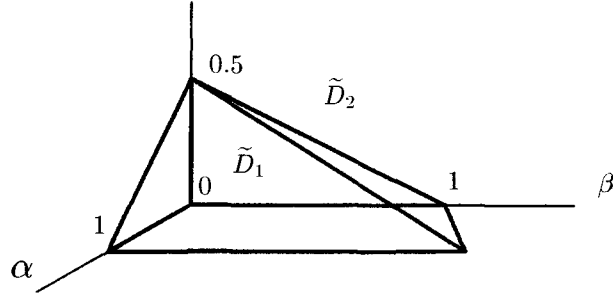
Fig. 4. Estimate of the regular-dynamics domain for inhomogeneous chains with two element types in the parameter space $(\alpha, \beta, \varepsilon)$.

where R'_i is either the row quasi-norm of the matrix A, $R'_i = \sum_{j=1, j \neq i}^{n} |a_{ij}|$, or the columnar quasi-norm of the matrix A, $R'_j = \sum_{i=1, i \neq j}^{n} |a_{ij}|$. Because map differential (12) is a real symmetric matrix, the row and the columnar quasi-norms coincide, and the eigenvalues of D_f are real. Therefore, the Guershgorin circles are transformed into intervals on the real axis:

$$\rho_s \in \bigcup_{i=1}^{n} \{ z \in \mathbb{R} : |z - a_{ij}| \leq R'_i \}, \quad s = 1, 2, \ldots, N.$$

Because $\varepsilon > 0$, it is easy to find from expression (12) that

$$R'_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| = |\varepsilon| + |\varepsilon| = 2\varepsilon.$$

We note that R'_i does not depend on i and there are only two different types of diagonal elements in the map differential D_f (see Eq. (12)); therefore, the eigenvalues of D_f belong to the union of two intervals

$$\{ z \in \mathbb{R} : |1 - 2\varepsilon - 2\alpha - z| \leq 2\varepsilon \} \cup \{ z \in \mathbb{R} : |1 - 2\varepsilon - 2\beta - z| \leq 2\varepsilon \}.$$

Because $\alpha, \beta, \varepsilon > 0$, we then obtain an upper estimate for the absolute values of the eigenvalues of D_f:

$$|\rho_s| \leq \begin{cases} \beta + 2\varepsilon, & \alpha < \beta, \\ \alpha + 2\varepsilon, & \alpha > \beta. \end{cases}$$

Therefore, the domain $\widetilde{D}_1$ in the parameter space $(\alpha, \beta, \varepsilon)$ satisfying the conditions

$$\widetilde{D}_1 : \quad \begin{cases} \beta + 2\varepsilon < 1 & \text{for } \alpha < \beta, \\ \alpha + 2\varepsilon < 1 & \text{for } \alpha > \beta, \end{cases} \tag{13}$$

corresponds to a regular dynamics of inhomogeneous model (11), (2), and the domain

$$\widetilde{D}_2 : \quad \begin{cases} \beta + 2\varepsilon > 1 & \text{for } \alpha < \beta, \\ \alpha + 2\varepsilon > 1 & \text{for } \alpha > \beta, \end{cases} \tag{14}$$

corresponds to positive λ_s. The domains $\widetilde{D}_1$ and $\widetilde{D}_2$ are shown in Fig. 4. The domain $\widetilde{D}_1$ is a *lower* estimate for the region of regular dynamics, because it was derived using the *upper* estimate for the eigenvalues of
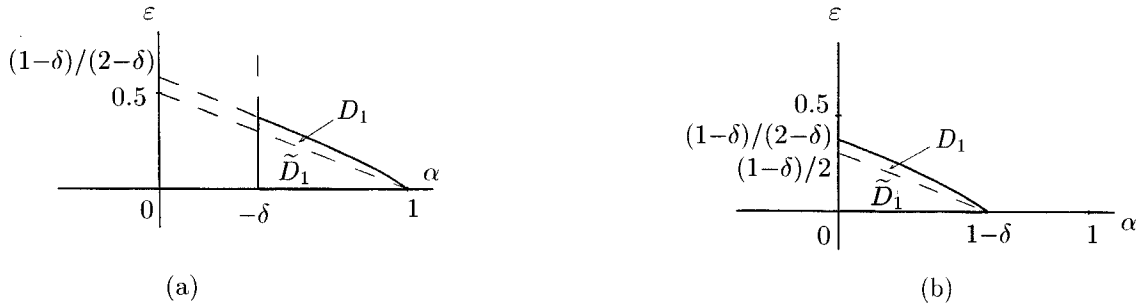
Fig. 5. Approximate estimate and exact result for the regular-dynamics domain of the model with periodic inhomogeneity: (a) for $-1 < \delta < 0$; (b) for $0 < \delta < 1$.

the map differential. Any point of the parameter space belonging to $\widetilde{D}_1$ corresponds to a regular dynamics of the model, while any point belonging to $\widetilde{D}_2$ corresponds to either regular or (if the motion is finite) chaotic dynamics.

The derived estimate is valid not only for the single-defect model discussed but also for *the entire class* of annular systems of maps with a diffusive concatenation of form (1) characterized by the presence of two types of the elements (with different parameters α_i of the function $G(x_i, \alpha_i, \gamma)$). In this case, the number of elements of each type and their relative positions *do not matter*. Indeed, the quasi-norms of the map differentials for all such systems coincide, and the centers of two possible intervals containing the eigenvalues of D_f (according to the Guershgorin theorem) are common as well.

By a corollary of the Guershgorin theorem, if the two intervals where the estimate of ρ_s is made do not intersect, then there is a "clustering" of the map differential eigenvalues. That is, q eigenvalues ρ_s, $s = 1, 2, \ldots, q$, lie within one interval, and $N-q$ eigenvalues ρ_s, $s = q + 1, q + 2, \ldots, N$, lie within the other interval. Here, q is the number of elements with the parameter α, and $N-q$ is the number of elements with the parameter β. However, taking this property into account does not improve the estimate of the regular-dynamics domain.

We compare estimate (13) with the exact result obtained above for the annular model with a spatially periodic inhomogeneity (see Sec. 2.2). It is clear that the model belongs to the class for which this estimate remains valid. For elucidation, we construct the sections $\delta = \text{const}$ of the domain D_1 corresponding to a regular dynamics of system (3), (2) and also those for the estimate $\widetilde{D}_1$ (Fig. 5). It can be seen in the figure that estimate (13) for the regular-dynamics domain approximates the true behavior of the model rather well. However, the estimate fails to reflect that for $-1 < \delta < 0$, the regular-dynamics domain for system (3), (2) is broader than the regular-dynamics domain of the spatially homogeneous system.

We now return to system of maps (11), (2) whose spatial inhomogeneity is connected with the presence of only one defect. We find the eigenvalues of map differential (12) *numerically*. In this case, the characteristic polynomial can be written as

$$\det(Q_N - \rho I_N) = \begin{vmatrix} 2z\varepsilon + 2(\alpha - \beta) & \varepsilon & 0 & \cdots & \varepsilon \\ \varepsilon & 2z\varepsilon & \varepsilon & \cdots & 0 \\ 0 & \varepsilon & 2z\varepsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & 0 & 0 & \cdots & 2z\varepsilon \end{vmatrix},$$

where $2z\varepsilon = 1 - 2\varepsilon - 2\alpha - \rho$. Expanding $\det(Q_N - \rho I_N)$ with respect to the first row, we obtain

$$\det(Q_N - \rho I_N) = \left[2z\varepsilon + 2(\alpha - \beta)\right] B_{N-1} - 2\varepsilon^2 B_{N-2} - 2(-1)^N \varepsilon^N, \tag{15}$$
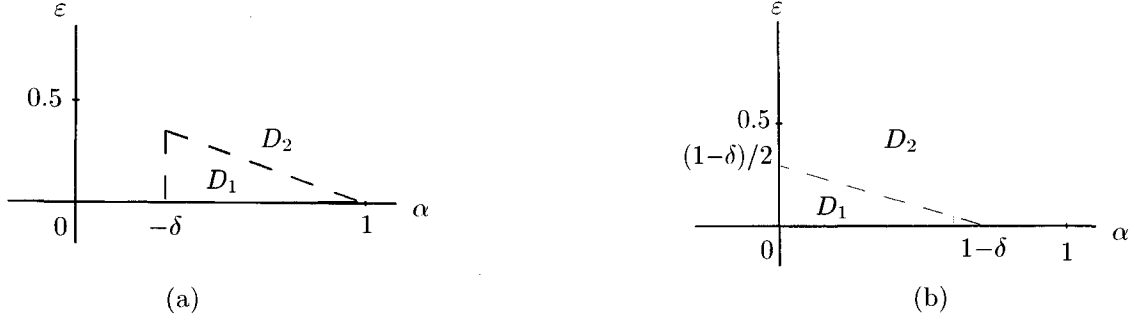
Fig. 6. Characteristic form of the sections δ = const of the parameter space for a chain of $N=20$ elements with a single defect: (a) for $\delta = -0.3$; (b) for $\delta = 0.3$.

where

$$B_N = \begin{vmatrix} 2z\varepsilon & \varepsilon & 0 & \cdots & 0 \\ \varepsilon & 2z\varepsilon & \varepsilon & \cdots & 0 \\ 0 & \varepsilon & 2z\varepsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2z\varepsilon \end{vmatrix}.$$

As above, we use the expansion of B_N with respect to the first row to obtain the recursive relation $B_N = 2z\varepsilon B_{N-1} - \varepsilon^2 B_{N-2}$. This relation can be interpreted as a difference equation with the initial conditions $B_1 = 2z\varepsilon$, $B_2 = \varepsilon^2(4z^2 - 1)$, whose solution is of the form

$$B_N = \varepsilon^N U_N(z), \tag{16}$$

where $U_N(z)$ is a second-kind Chebyshev polynomial. Taking Eq. (16) and the expression for $U_N(z)$,

$$U_N(z) = \frac{\left(z + \sqrt{z^2 - 1}\right)^{N+1} - \left(z - \sqrt{z^2 - 1}\right)^{N+1}}{2\sqrt{z^2 - 1}} = \frac{(1/t)^{N+1} - t^{N+1}}{1/t - t},$$

where $t = z - \sqrt{z^2 - 1}$, into account, we obtain

$$\det(Q_N - \rho I_N) = 2\varepsilon^N \left[\frac{t^{2N+1} + 1}{2t^N} + \left(\frac{\alpha - \beta}{\varepsilon} \right) \frac{1 - t^{2N}}{(1 - t^2)\, t^{N-1}} - (-1)^N \right]$$

from Eq. (15). Hence, the characteristic equation for map differential (12) can be expressed as

$$t^{2N+2} + 2\left(\frac{\alpha - \beta}{\varepsilon} \right) t^{2N+1} - t^{2N} - 2(-1)^N t^{N+2} + +2(-1)^N t^N + t^2 - 2\left(\frac{\alpha - \beta}{\varepsilon} \right) t - 1 = 0. \tag{17}$$

It is now easy to calculate ρ_s by finding the roots of Eq. (17) numerically. Comparing the ρ_s values to unity for all possible values of the parameters α, β, and ε, it is straightforward to determine the domains D_1 and D_2 of qualitatively different dynamics for an annular chain of diffusively connected maps with a single defect. The sections of these domains by the planes δ = const (where $\delta = \beta - \alpha$) are shown in Fig. 6. Estimate (13) for the regular-dynamics domain is shown by the dashed line. In this case as it was for the model with a periodic inhomogeneity, this domain (for positive δ) is broader than that predicted by the estimate. However, in contrast to the periodic inhomogeneity, the estimate for $\delta < 0$ fully coincides with the numerical result and also with the result for a homogeneous chain. Therefore, if the parameter corresponding to one of the system elements is smaller than those corresponding to all other elements, then this defect has no influence on the dynamics of the entire ensemble. However, if the anomalous parameter value is larger than the normal value, then the regular dynamics domain of the model is narrower than that in the homogeneous case.

3. Conclusion

We investigated the behavior of systems of diffusively concatenated one-dimensional maps that model a one-dimensional active medium with a spatial inhomogeneity. Different types of inhomogeneity were examined. For all systems whose spatial inhomogeneity is related to the presence of two types of elements that differ by a parameter value, a lower estimate for the regular ensemble dynamics domain in the space of the parameters was derived *analytically*. This estimate was based on the localization of the differential eigenvalues of the map. Numerical and exact analytic results were obtained for two specific models of this class.

The results for the three examined cases, i.e., estimate (13) of the regular-dynamics domain for the class of models with two types of elements, the exact results for the system with periodic inhomogeneity (Fig. 3), and the exact results for the single-defect model (Fig. 6), reveal that there exist two general types of system dynamics. For the first type, the parameter of one sort of elements belongs to the regular-dynamics domain while the parameter of the other sort belongs to the unstable-behavior domain, and the general dynamics remains regular. For the second type, in contrast, the model dynamics can be unstable or, in the case of finite motion, chaotic. The realization of either behavior type depends on the diffusion parameter ε and is possible only if the parameter is small. Obviously, the general dynamics of all chains belonging to the class of our interest cannot be regular for sufficiently large values of ε.

As far as we know, this work is the first attempt to describe the dynamics of *inhomogeneous* systems of concatenated maps analytically. The present knowledge is far from exhaustive even for the homogeneous discrete models of continuous media. Much is yet to be elucidated, many qualitative and quantitative questions remain open. Moreover, this work did not address the problems of possible stationary states of the model in the regularity domain, of wave motions along the chain and their dependence on the defects, of the dynamics in the case of time-dependent parameters, etc. Some of these questions will be answered in the near future [23, 24].

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