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Dynamical chaos: systems of classical mechanics

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Contents

Abstract. This article is a methodological manual for those who are interested in chaotic dynamics. An exposition is given on the foundations of the theory of deterministic chaos that originates in classical mechanics systems. Fundamental results obtained in this area are presented, such as elements of the theory of nonlinear resonance and the Kolmogorov – Arnol'd – Moser theory, the Poincaré – Birkhoff fixed-point theorem, and the Mel'nikov method. Particular attention is given to the analysis of the phenomena underlying the self-similarity and nature of chaos: splitting of separatrices and homoclinic and heteroclinic tangles. Important properties of chaotic systems — unpredictability, irreversibility, and decay of temporal correlations — are described. Models of classical statistical mechanics with chaotic properties, which have become popular in recent years — billiards with oscillating boundaries — are considered. It is shown that if a billiard has the property of well-developed chaos, then perturbations of its boundaries result in Fermi acceleration. But in nearly-integrable billiard systems, excitations of the boundaries lead to a new phenomenon in the ensemble of particles, separation of particles in accordance their velocities. If the initial velocity of the particles exceeds a certain critical value characteristic of the given billiard geometry, the particles accelerate; otherwise, they decelerate.

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1. Introduction

For a long time, the concept of chaos was associated with the assumption that, at least, the excitation of an extremely high number of degrees of freedom is necessary in the system. The formation of this idea seems to have been influenced by the concepts of statistical mechanics, in which the motion of an individual gas particle can be predicted in principle but the behavior of a system consisting of a huge number of particles is extremely complex, and therefore a detailed dynamic approach is meaningless. This dictated the need for a statistical analysis. But extensive studies have demonstrated that the validity of the statistical laws and statistical description is not restricted to highly complex systems with a large number of the degrees of freedom. Random behavior can also be exhibited by entirely deterministic systems with a moderate number of the degrees of freedom. Here, the point is not the complexity of the system or the presence of external noise but the emergence of an exponential instability of motion at certain values of the parameters. The dynamics of systems subject to such an instability is called dynamic stochasticity, or deterministic (dynamic) chaos. Investiga-

tions in this area are of fundamental importance because they disclose the nature of randomness by supplementing the hypothesis of molecular chaos with the hypothesis of dynamic stochasticity.

H Poincaré [1] was the first to note the relation between statistics and instabilities. However, a statistical approach to the description of systems with numerous degrees of freedom was previously suggested by L Boltzmann [2], who conjectured that the motion of particles in a rarefied gas should be regarded as random and that the entire energetically allowed phase-space domain is accessible to each particle. Such a view of the properties of multiparticle systems, known as the ergodic hypothesis [2–4], became the basis of classical statistical mechanics. A rigorous substantiation of this hypothesis could not be found for a long time, however. Some progress in this direction was achieved due to studies by P Ehrenfest [5, 6] (see also Refs [7, 8]). In particular, they allowed establishing applicability limits for the laws of statistical mechanics. However, a well-known study by E Fermi, J Pasta, and S Ulam [9] (see Refs [10, 11] for a more detailed exposition), who made the first attempt to verify the ergodicity hypothesis, put the problem of substantiation of statistical physics in the forefront again.

This problem can be partially resolved based on studies by Poincaré (see Ref. [12]), who concluded that the motion of a system is extremely complex in the neighborhood of unstable fixed points in phase space. This was the earliest indication of the possible chaotic properties of nonlinear dynamical systems. Later, G D Birkhoff [13] showed that if the ratio of the frequencies is rational (i.e., at a resonance), stable and unstable fixed points appear in phase space. Higher-order resonances change the topology of phase trajectories and lead to the formation of a chain of 'islands.' It turned out that the regular perturbation theory fails to describe such resonances, because the solutions are strongly perturbed near the resonances, and therefore small denominators emerge in the expansion and the series diverge.

N S Krylov [14] was the first to deeply investigate the nature of statistical laws. He showed that the property of mixing and the related local instability of nearly all trajectories of the corresponding dynamical systems underlie this nature. In view of this, M Born [15] (see also [16]) suggested that the behavior of the systems is not predictable in classical mechanics. Later, the dynamics due to such instabilities in the systems came to be known as dynamic stochasticity, or deterministic (dynamic) chaos. The word 'chaos,' in this meaning, seems to have been introduced by J A Yorke [17] (see Ref. [18], p. 338). But as noted by Ya G Sinai, the word combination 'deterministic chaos' was first used by B Chirikov and G Ford in the 1960s.

Physically, due to unavoidable fluctuations (i.e., small perturbations of the initial conditions), the initial state of the system is to be specified by some distribution. The problem is in predicting the evolution of the system based on this initial distribution. If the system is stable, such that small perturbations do not increase exponentially with time, its behavior is predictable. In contrast, if the system is subject to exponential instability (which is expressed by saying that the system has a sensitive dependence on the initial conditions), the process allows only a probabilistic description. In essence, precisely these considerations formed the basis of the modern views of dynamical chaos. The discovery that it is chaos, rather than external noise, that mainly determines the behavior of the system was unexpected (see Ref. [20] for a review).

The schools of A N Kolmogorov and A A Andronov, to which a brilliant group of outstanding contemporary mathematicians belongs, have deeply influenced the development of the theory of dynamical chaos. In particular, Kolmogorov's theorem of the preservation of almost periodic motion in weakly perturbed Hamiltonian systems, proved by V I Arnol'd and J Moser and known as the Kolmogorov–Arnol'd–Moser (KAM) theorem (see Refs [21–25]), became a keystone in understanding the origin of chaotic behavior. In their early studies, D V Anosov [26] and Sinai [27, 28] showed that dynamical chaos is a widespread phenomenon.

In his pioneering investigations of the bifurcations of a saddle–focus separatrix [29, 30], L P Shil'nikov developed, among other things, a special technique for the analysis of the dynamics of systems near saddle-type trajectories and uncovered the extreme complexity of the structures that develop as homoclinic trajectories appear. It was demonstrated that the behavior of systems must be complex in the full neighborhood of the parametric values at which a homoclinic orbit exists. Later, Shil'nikov, L M Lerman, N K Gavrilov, I M Ovsyannikov, D V Turaev, and others developed new methods that allowed describing a finite number of bifurcations that lead to chaotic dynamics (see Refs [31, 32] and the references therein).

A new stage in explaining chaotic behavior and its origin in deterministic systems was initiated by Kolmogorov's and Sinai's studies [33–35], where the concept of entropy was introduced for dynamical systems. These studies laid the foundations of a consistent theory of chaotic dynamical systems.

Various abstract mathematical constructions have played an important role in the development of the theory of deterministic chaos. In particular, S Smale [36], to disprove the hypothesis of the density of systems that exhibit only a periodic-type behavior, constructed a notable example, currently known as the 'Smale horseshoe.' This example implies that there exist systems that have both an infinite number of periodic orbits with different periods and an infinite number of aperiodic trajectories [18, 36, 37]. Subsequent to the Smale horseshoe, Anosov's *C*-systems were found [26, 38], which are characterized by the most pronounced mixing properties. Such systems were generalized by introducing Smale's 'Axiom A' [37] (see also Refs [39–41] and the references therein) and hyperbolic sets [18, 37, 40–42]; these generalizations specified an important class of dynamical systems that have the property of the exponential instability of trajectories (see Ref. [43] for a review).

At nearly the same time, mathematical studies began appearing in which attempts were made to substantiate statistical mechanics based on the analysis of billiard systems [27, 28]. Originally, billiards were introduced as simplified models appropriate for the consideration of certain problems of statistical physics [13] (see also the references in Refs [44, 45]). A billiard on a plane is a dynamical system that describes bodies (balls) moving inertially inside a bounded domain, in accordance with the law of equality of the incidence and reflection angles. In essence, planar mathematical billiards are the usual billiards without friction, although with an arbitrary configuration of the table and without pockets.

Krylov's problem of mixing in a system of elastic balls [14] was first solved using billiard systems. Furthermore, it was shown that systems corresponding to billiards with scattering

boundaries have much in common with geodesic flows in negative-curvature spaces, i.e., with Anosov flows. Somewhat later, the class of billiard systems capable of exhibiting chaotic properties was substantially extended (see Refs [45–47] and the references therein). It was shown based on a generalization of such systems (a modified two-dimensional Lorentz gas) that motion in purely deterministic systems can be similar to Brownian motion [44, 45]. This result became the first rigorous confirmation of chaotic properties exhibited by dynamical systems (not involving any random mechanism).

Further investigations of nonlinear systems, both theoretical and experimental, have shown how typical the chaotic behavior is in systems with few degrees of freedom. It became obvious that chaotic properties can be manifested by a variety of nonlinear systems; if chaos is not revealed, this can merely result from the fact that the development of chaos is restricted either to very small domains in the parameter space or to physically unrealizable domains.

How does chaotic motion originate? What is the nature of chaos? Seemingly, there should be many paths toward the onset of chaos. But it became known that the scenarios of chaotization are far from numerous. Moreover, some of them obey universal laws and are independent of the nature of the system. The same development scenarios for chaos are inherent in a variety of objects. The universal behavior resembles the usual second-order phase transitions, and invoking renormalization-group and scaling techniques known in statistical mechanics opens new avenues in investigating chaotic dynamics.

This article presents the foundations of the theory of dynamical chaos. We describe the principal results obtained in the field that belong to classical mechanics, such as elements of the theory of nonlinear resonance and the KAM theory; the Poincaré–Birkhoff fixed-point theorem, which is important for understanding the sources of chaotic behavior; and the Mel'nikov method, which allowed obtaining a criterion for the origin of chaos analytically in some cases. Particular attention is given to the nature of chaos. Specifically, we detail the factors that lead perturbed systems to manifest self-similarity, to the splitting of separatrices, and to homoclinic and heteroclinic tangles. We also show that unpredictability, irreversibility, and decay of temporal correlations occur in systems in which such phenomena are observed.

In Section 9, we describe models of nonequilibrium classical statistical mechanics with chaotic properties, highly popular today: billiards with oscillating boundaries. The Lorentz gas and stadium-type billiards are considered in detail. An interesting result is presented: the analytic form of billiard-particle acceleration law, i.e., a proof of the presence of Fermi acceleration in billiards with well-developed chaos. But if a billiard system is near an integrable system, such that the curvature of the billiard boundary is not large, small oscillations of the boundary lead to a new phenomenon. A specific, billiard version of Maxwell's demon originates: a billiard particle either accelerates or decelerates depending on the initial conditions. In other words, perturbation of the boundaries of such billiards results in a velocity stratification of the particle ensemble.

The modern mathematical techniques used to analyze the chaotic properties of dynamical systems are fairly complex. But in this article, we pursue the aim of giving a general idea of the origin of the phenomenon of deterministic chaos and expose the fundamental concepts underlying the currently known approaches to chaotic dynamics. Therefore, our presentation is mainly based on geometric techniques and a qualitative approach. Although most of the results described here have been known for a rather long time, we present them in a form appropriate for a nonexpert to comprehend the origins of chaotic dynamics. Therefore, in particular, this article is methodological.

2. Background

The subject of our analysis is the systems described by ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, a)\,, \tag{1}$$

where $\mathbf{v} = \{v_1, v_2, \ldots, v_l\}$ is a vector function (typically assumed to be smooth), a symbolizes the set of parameters, and $\mathbf{x} = \{x_1, x_2, \ldots, x_l\}$ is an l-dimensional vector with components $x_1, x_2, \ldots, x_l$ that characterizes the state of a dynamical system. If the substitution of a function $\mathbf{F}(t, c_i)$, $c_i = \text{const}$, in Eqn (1) turns them into identities, this function is called a solution. Specifying initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ uniquely determines the solution at any instant t,

$$\mathbf{x}(t) = \mathbf{F}(t, \mathbf{x}_0)\,. \tag{2}$$

Any solution $\mathbf{x}(t)$ of system of equations (1) can be geometrically represented by a curve in the l-dimensional space of variables $x_1, \ldots, x_l$. This l-dimensional space is called the phase space of the system (we let M denote this space). Each state of the dynamical system corresponds to a point in M, and each point in M corresponds to a unique state of the system. Changes in the state of the system can be interpreted as the motion of a point (called the representation point) in the phase space. The trajectory of this representation point, i.e., the set of its consecutive positions in the phase space M, is called the phase trajectory.

We assume that system (1), whose phase space is M, was in the state $\mathbf{x}_0$ at an instant t_0. Then, generally, it is in another state at an instant $t \neq t_0$. We let $\mathbf{F}^t\mathbf{x}_0$ denote this new state. Thus, for any t, we define the evolution operator, or the shift map $\mathbf{F}^t : M \to M$ of the phase space M into itself. The map $\mathbf{F}^t$ transforms the system from its state at t_0 into the state at t. In other words, solution (2) of Eqns (1) establishes a correspondence between the point $x_i(t_0)$, $i = 1, 2, \ldots, l$, of the phase space M at the instant t_0 and a certain phase-space point $x_i(t)$ at an instant t:

$$\mathbf{F}^t\mathbf{x}_0 = \mathbf{x}(t)\,. \tag{3}$$

Therefore, in the general case, any phase-space domain Ω_0 transforms in time t into another domain, $\Omega_t = \mathbf{F}^t\Omega_0$, under the action of the map $\mathbf{F}^t$. The map $\mathbf{F}^t : M \to M$ is also called the phase flow, and the function $\mathbf{v}(\mathbf{x})$ is the vector field of the given dynamical system whose phase space is M.

If the phase flow $\mathbf{F}^t$ has a secant S, i.e., a certain codimension-1 hypersurface that is transversally intersected by phase curves, then a map Φ can be defined on S as follows: to any point p of S we associate the closest (next to p) point, p', of intersection of the phase curve with the same hypersurface S. Then the analysis of the dynamics of the original system reduces to analyzing the properties of the map Φ, which is called the succession function, or the Poincaré map (sometimes, the Poincaré return map).

Systems for which $\operatorname{div}\mathbf{v}=0$ are called conservative. Most objects considered in classical mechanics belong to this class. Investigating the evolution of conservative systems is of fundamental importance because it is related to a number of problems such as the substantiation of the Boltzmann ergodic hypothesis, planetary motion and the N-body problem, the dynamics of charged particles and plasma heating, etc.

We begin our analysis with the Hamiltonian approach, whose advantage is not in its formalism but in a deeper interpretation of the physical essence of the phenomenon in question. In fact, the Hamiltonian approach is a geometrical method of analysis. It has some advantages and allows obtaining solutions to problems that are not amenable to other techniques of analysis.

3. Hamiltonian mechanics

The Hamiltonian formalism is based on the well-known Hamilton equations, which are given by the system of ordinary differential equations

$$\dot{q}_i=\frac{\partial H}{\partial p_i}\,,\qquad \dot{p}_i=-\frac{\partial H}{\partial q_i} \tag{4}$$

(where $i=1,2,\ldots,n$), and for which the initial conditions at $t=t_0$ are specified as $q_i(t_0)=q_i^0$, $p_i(t_0)=p_i^0$. Such systems are rich in a variety of motions, from completely integrable dynamics to quasi-periodicity and chaos.

Among the fundamental properties of Hamiltonian systems is the conservation of the volume of an arbitrary phase-space domain, i.e., the validity of the Liouville theorem,

$$\int_{D_0}\mathrm{d}\mathbf{q}_0\,\mathrm{d}\mathbf{p}_0=\int_{D_t}\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\,,$$

where $D_0, D_t\subset M$.

3.1 Integrable systems

The problem of the integrability of Hamiltonian systems is quite complex. There are a number of rather general (although, naturally, not universal) methods that in some cases allow constructing a solution of Eqns (4) or of an approximation to them. Quite comprehensive reviews and monographs describing these methods in detail are available (see, e.g., Refs [48 – 55] and the references therein). Therefore, we do not consider them here; instead, we only present a geometric analysis of integrable systems.

Hamiltonian system (4) is completely integrable (and the Hamiltonian H is integrable) if there exists a canonical transformation to angle – action variables, $\mathbf{q},\mathbf{p}\to\boldsymbol{\alpha},\mathbf{J}$. Another definition is based on the Liouville theorem on integrable systems: a Hamiltonian system with n degrees of freedom is integrable if n independent integrals in involution are known for this system.

Systems with one degree of freedom ($n=1$) are always integrable because their Hamiltonian $H(q,p)=E$ is an integral of motion. A pendulum model is a highly representative example of such systems. Its Hamiltonian can be written as

$$H=\frac{p^2}{2ml^2}-mgl\cos\varphi\,, \tag{5}$$

where φ is the angle of deviation from the vertical and g is the acceleration of gravity. The equations of motion of the pendulum are $\dot{p}=-mgl\sin\varphi$, $\dot{\varphi}=p/ml^2$, or

$$\ddot{\varphi}+\omega_0^2\sin\varphi=0\,, \tag{6}$$

where $\omega_0=\sqrt{g/l}$ is the oscillation frequency. Hamiltonians of this type are typical of many problems and play a fundamental role in classical mechanics.

If the full energy of the pendulum, $H=E$, exceeds the maximum value of the potential energy, $E=E_{\mathrm{rot}}>mgl$, the momentum p is always different from zero, which leads to an infinite increase in the angle φ, i.e., to the rotation of the pendulum. Energies $E=E_{\mathrm{osc}}<mgl$ correspond to oscillations of the pendulum. If $E\approx E_{\mathrm{s}}\equiv mgl$, the oscillation period tends to infinity and the motion follows the separatrix between the two types of motion, oscillation and rotation (Fig. 1a). The equation of the separatrix can be written as $p_{\mathrm{s}}=\pm 2\omega_0 ml^2\cos\varphi_{\mathrm{s}}/2$, $\varphi_{\mathrm{s}}=4\arctan\left[\exp\left(\omega_0 t\right)\right]-\pi$, where the plus and minus signs respectively correspond to the upper and lower branches.

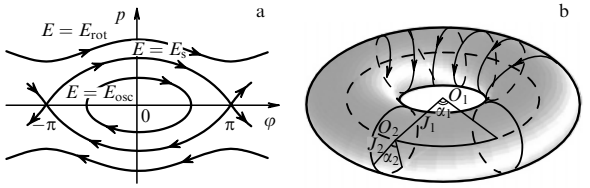

Figure 1. (a) Phase portrait of a nonlinear pendulum and (b) a visual representation of an integrable Hamiltonian system with two degrees of freedom in angle – action variables.

In the neighborhood of the points with the coordinates $(p,\varphi)=(0,2\pi k)$, $k=0,\pm1,\pm2,\ldots$, the family of phase curves consists of ellipses. Such points are therefore called elliptic. The family of trajectories near the points $(p,\varphi)=(0,\pi+2\pi k)$, $k=0,\pm1,\pm2,\ldots$, is formed by hyperbolas, and such points are called hyperbolic.

For an autonomous system with two degrees of freedom, $n=2$, the integrable system in the angle – action variables $(\mathbf{J},\boldsymbol{\alpha})$ has the topology of a two-dimensional torus (Fig. 1b). In view of the constraints $\omega_1=\omega_1(J_1,J_2)$ and $\omega_2=\omega_2(J_1,J_2)$ (if they are present), the frequencies of gyration in circles O_1 and O_2 for nonlinear systems can vary from torus to torus. Their ratio can also vary:

$$\frac{\omega_1}{\omega_2}=\frac{\omega_1(J_1,J_2)}{\omega_2(J_1,J_2)}\,. \tag{7}$$

If quantity (7) is rational, $\omega_1/\omega_2=k/m$ (a resonance), the dynamics of the system are periodic: the phase trajectory closes after k windings over the circle O_1 and m windings over the circle O_2. If fraction (7) is irrational, $\omega_1/\omega_2\neq k/m$, the phase trajectory covers the torus everywhere densely, and the motion of the system is called quasi-periodic, of almost periodic.

Therefore, because J_1 and J_2 are arbitrary, the phase space is represented by two-dimensional tori, which can be visualized in ordinary three-dimensional space as a set of tori nested in one another, with the major and minor semiaxes specified by the ratios J_1 and J_2 (Fig. 2).

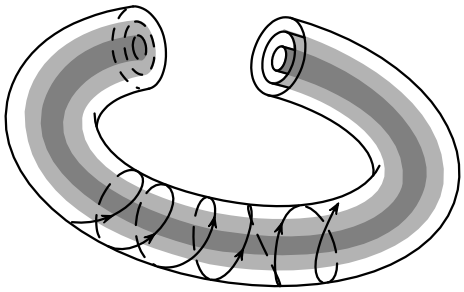

Figure 2. General structure of the phase space of an integrable system with two degrees of freedom in the action – angle variables.

For integrable systems with n degrees of freedom, their phase space is $2n$-dimensional; in the angle – action variables, it has the topology of a set of n-dimensional tori. Any possible trajectory lies on one of them. Some trajectories may be closed; others cover the corresponding torus everywhere densely.

A torus of dimension $n \geqslant 2$ with given $J_1, J_2, \ldots, J_n$ is called resonant if the relation

$$\sum_{i=1}^{n} k_i \omega_i(J_1, J_2, \ldots, J_n) = 0\,,$$

where k_i are nonzero integers, holds for a set of frequencies $\{\omega_i(J_1, J_2, \ldots, J_n)\}$ $(i = 1, 2, \ldots, n)$.

3.2 Perturbed motion

The overwhelming majority of Hamiltonian equations (for systems with n degrees of freedom) are not integrable. But a Hamiltonian $H = H(\boldsymbol{\alpha}, \mathbf{J})$ can in some cases be divided into an integrable part $H_0 = H_0(\mathbf{J})$ and a part that is not integrable but can be represented as a small perturbation H_1 of H_0:

$$H(\boldsymbol{\alpha}, \mathbf{J}) = H_0 + \varepsilon H_1\,, \tag{8}$$

where $H_0 = H_0(\mathbf{J})$, $H_1 = H_1(\boldsymbol{\alpha}, \mathbf{J})$, and ε is a small parameter. Systems whose Hamiltonians can be written in form (8) are called nearly integrable systems.

In the chosen variables $\boldsymbol{\alpha}$ and $\mathbf{J}$, the canonical equations that follow from Eqns (4) and (8) have the form

$$\dot{J}_i = -\varepsilon \frac{\partial H_1}{\partial \alpha_i}\,, \qquad \dot{\alpha}_i = \frac{\partial H_0}{\partial J_i} + \varepsilon \frac{\partial H_1}{\partial J_i}\,, \qquad i = 1, 2, \ldots, n\,, \tag{9}$$

where $\varepsilon \ll 1$. If $\varepsilon = 0$, system (9) is completely integrable, and its solutions cover n-dimensional tori. We now assume that $\varepsilon \neq 0$. How strong are then the changes in the character of the integrable system?

4. Nonlinear resonance

The answer to the question posed in Section 3.2 essentially depends on the relation between the perturbing force frequency and the proper frequency of the system. If the proper frequency is close to the frequency of the external force, this results in an increase in amplitude, leading to a resonance. But for nonlinear systems with perturbation (8), the frequency depends on the amplitude, and hence the system falls out of resonance some time later. This reduces the amplitude, which, in turn, changes the frequency. Therefore, the system shortly returns to the neighborhood of the resonance. The so-called phase oscillations thus set in.

Resonances can arise not only between the system and external influences but also between different degrees of freedom of the system itself, which corresponds to the autonomy of the Hamiltonian H_1 in Eqn (8). This is the case of so-called internal resonances.

If the resonance is not isolated, the overlapping of resonances results in a very complex motion in the system. Moreover, the resonances prevent finding solutions of the equations using the technique of the canonical perturbation theory. In the perturbation theory, the original system is approximated by a close integrable system subject to a small perturbation, and the solution is sought as an expansion in powers of ε. The presence of resonances disrupts the convergence of such series, because this approach implicitly assumes that the original equations are integrable. This is not the case in most situations, however. Even very simple systems may not be integrable, and their dynamics may be very complex at certain initial conditions. For example, signatures of dynamical chaos are manifested in the behavior of a nonlinear pendulum [56, 57]. The perturbation theory cannot describe such a complex behavior, which is formally reflected by the divergence of the series. If the initial conditions of the system correspond to regular trajectories (quasi-periodic motion), such trajectories undergo qualitative rearrangements under the action of perturbations in the neighborhood of the resonances.

The theory of nonlinear resonance is remarkable for its ability to obtain an analytic criterion for the onset of irregular motion in a Hamiltonian system. This criterion was originally introduced in [58, 59]. We consider the theory of nonlinear resonance from a more general standpoint, following Refs [60, 61] (see also Refs [54, 62]). A more complete exposition of the theory of resonances is given in [63].

4.1 Small denominators

We first consider the internal resonance. We decompose the function $H_1(\boldsymbol{\alpha}, \mathbf{J})$ [see Eqn (8)] into a Fourier series, $H_1(\boldsymbol{\alpha}, \mathbf{J}) = \sum_k H_1^k(\mathbf{J}) \exp(\mathrm{i}\mathbf{k}\boldsymbol{\alpha})$, and substitute this decomposition in Hamilton equations (9). This yields

$$\begin{aligned} \dot{J}_j &= -\mathrm{i}\varepsilon \sum_k k_j H_1^k \exp(\mathrm{i}\mathbf{k}\boldsymbol{\alpha})\,, \\ \dot{\alpha}_j &= \omega_j(\mathbf{J}) + \varepsilon \sum_k \frac{\partial H_1^k}{\partial J_j} \exp(\mathrm{i}\mathbf{k}\boldsymbol{\alpha})\,, \qquad j = 1, 2, \ldots, n\,, \end{aligned} \tag{10}$$

where $\mathbf{k}$ is a vector with real integer components, H_1^k are the Fourier coefficients, and $\omega_j(\mathbf{J}) = \partial H_0(\mathbf{J})/\partial J_j$.

We seek a solution of perturbed system (10) in the form of a series in powers of the small parameter ε:

$$\begin{aligned} J_j &= J_j^{(0)} + \sum_{s=1}^{\infty} \varepsilon^s J_j^{(s)}\,, \\ \alpha_j &= \alpha_j^{(0)} + \sum_{s=1}^{\infty} \varepsilon^s \alpha_j^{(s)}\,, \qquad j = 1, 2, \ldots, n\,. \end{aligned} \tag{11}$$

Next, using expansion (11) and Eqns (10), we select the terms containing equal powers of ε. In the zeroth approximation, we then find $\dot{J}_j^{(0)} = 0$, $\dot{\alpha}_j^{(0)} = \omega_j(\mathbf{J}^{(0)})$, $j = 1, 2, \ldots, n$. Therefore,

$$J_j^{(0)} = \text{const}\,, \qquad \alpha_j^{(0)} = \omega_j(\mathbf{J}^{(0)})t + \text{const}\,. \tag{12}$$

Because

$$\omega_j\bigl(\mathbf{J}^{(0)}+\varepsilon\mathbf{J}^{(1)}\bigr)\approx\omega_j\bigl(\mathbf{J}^{(0)}\bigr)+\varepsilon\sum_m\frac{\partial\omega_j(\mathbf{J}^{(0)})}{\partial J_m^{(0)}}\,J_m^{(1)}\,,$$

we can easily obtain the first-approximation equations

$$\begin{aligned}\dot J_j^{(1)}&=-\mathrm{i}\sum_k k_j\tilde H_1^k(\mathbf{J}^{(0)})\exp\bigl(\mathrm{i}\mathbf{k}\boldsymbol{\omega}(\mathbf{J}^{(0)})t\bigr)\,,\\ \dot\alpha_j^{(1)}&=\sum_m\frac{\partial\omega_j(\mathbf{J}^{(0)})}{\partial J_m^{(0)}}\,J_m^{(1)}+\sum_k\frac{\partial\tilde H_1^k(\mathbf{J}^{(0)})}{\partial J_j}\exp\bigl(\mathrm{i}\mathbf{k}\boldsymbol{\omega}(\mathbf{J}^{(0)})t\bigr)\,,\end{aligned}\tag{13}$$

where $\tilde H_1^k$ are the values of the coefficients H_1^k with the constant taken into account. Equations (13) can be straightforwardly integrated:

$$\begin{aligned}J_j^{(1)}&=-\sum_k\frac{k_j\tilde H_1^k(\mathbf{J}^{(0)})}{\mathbf{k}\boldsymbol{\omega}}\exp(\mathrm{i}\mathbf{k}\boldsymbol{\omega}t)+\mathrm{const}\,,\\ \alpha_j^{(1)}&=\mathrm{i}\sum_m\frac{\partial\omega_j(\mathbf{J}^{(0)})}{\partial J_m^{(0)}}\sum_k\frac{k_m\tilde H_1^k(\mathbf{J}^{(0)})}{(\mathbf{k}\boldsymbol{\omega})^2}\exp(\mathrm{i}\mathbf{k}\boldsymbol{\omega}t)\\ &\quad-\mathrm{i}\sum_k\frac{\partial\tilde H_1^k(\mathbf{J}^{(0)})}{\partial J_j}\,\frac{\exp(\mathrm{i}\mathbf{k}\boldsymbol{\omega}t)}{\mathbf{k}\boldsymbol{\omega}}\,.\end{aligned}\tag{14}$$

It can easily be seen that if the condition

$$\mathbf{k}\boldsymbol{\omega}=k_1\omega_1+k_2\omega_2+\ldots+k_n\omega_n\approx0\tag{15}$$

(called the resonance relation) holds, terms with zero or near-zero denominators appear in Eqns (14). This leads to a substantial increase in the corrections $\alpha_j^{(s)}$ and $J_j^{(s)}$, which obviously disrupts the convergence of series (11).

If the resonance relation is not satisfied in first-approximation equations (13), it can hold for equations of higher approximations, $s>1$. The resonance that manifests itself in the sth order of the perturbation theory is called the order-s resonance.

We now let the perturbation H_1 depend on time periodically (with a period $T=2\pi/\Omega$), i.e., $H_1(\boldsymbol{\alpha},\mathbf{J},t)=H_1(\boldsymbol{\alpha},\mathbf{J},t+T)$. A treatment similar to that used in the preceding case yields

$$H_1(\boldsymbol{\alpha},\mathbf{J},t)=\sum_{k,m}H_1^{km}(\mathbf{J})\exp\bigl[\mathrm{i}(\mathbf{k}\boldsymbol{\alpha}-m\Omega t)\bigr]\,.$$

In this case, the Hamilton equations become

$$\dot J_j=-\mathrm{i}\varepsilon\sum_{k,m}k_jH_1^{km}\exp\bigl[\mathrm{i}(\mathbf{k}\boldsymbol{\alpha}-m\Omega t)\bigr]\,,$$

$$\dot\alpha_j=\omega_j(\mathbf{J})+\varepsilon\sum_{k,m}\frac{\partial H_1^{km}}{\partial J_j}\exp\bigl[\mathrm{i}(\mathbf{k}\boldsymbol{\alpha}-m\Omega t)\bigr],\quad j=1,2,\ldots,n\,.$$

As before, the zeroth-approximation solution is given by (12). The first-approximation equations can also be easily obtained and integrated, yielding

$$\begin{aligned}J_j^{(1)}&=-\sum_{k,m}\frac{k_j\tilde H_1^{km}(\mathbf{J}^{(0)})}{\mathbf{k}\boldsymbol{\omega}-m\Omega}\exp\bigl[\mathrm{i}(\mathbf{k}\boldsymbol{\omega}-m\Omega)t\bigr]+\mathrm{const}\,,\\ \alpha_j^{(1)}&=\mathrm{i}\sum_l\frac{\partial\omega_j(\mathbf{J}^{(0)})}{\partial J_l^{(0)}}\sum_{k,m}\frac{k_l\tilde H_1^{km}(\mathbf{J}^{(0)})}{(\mathbf{k}\boldsymbol{\omega}-m\Omega)^2}\exp\bigl[\mathrm{i}(\mathbf{k}\boldsymbol{\omega}-m\Omega)t\bigr]\\ &\quad-\mathrm{i}\sum_{k,m}\frac{\partial\tilde H_1^{km}(\mathbf{J}^{(0)})}{\partial J_i}\,\frac{\exp\bigl[\mathrm{i}(\mathbf{k}\boldsymbol{\omega}-m\Omega)t\bigr]}{\mathbf{k}\boldsymbol{\omega}-m\Omega}\,,\end{aligned}$$

where $\tilde H_1^{km}$ are the values of H_1^{km} with the inclusion of the constant that appears in the zeroth order. Thus, if the resonance relation $\mathbf{k}\boldsymbol{\omega}-m\Omega\approx0$ holds, perturbation-theory series (11) diverge; this is the essence of the small-denominator problem. To overcome this difficulty, it was suggested that a canonical transformation be used to pass to special (resonant) variables.

4.2 Universal Hamiltonian

Let the perturbation H_1 be a periodic function of time with a period $T=2\pi/\nu$ and let the motion be described by the Hamiltonian

$$H=H_0(J)+\varepsilon H_1(\alpha,J,t)\,,\tag{16}$$

where $\varepsilon\ll1$. We decompose the function H_1 into a Fourier series:

$$H_1(\alpha,J,t)=\sum_{k,m}H_1^{km}(J)\exp\bigl[\mathrm{i}(k\alpha-m\nu t)\bigr]\,.\tag{17}$$

Then Hamilton equations (9) become

$$\begin{aligned}\dot J&=-\mathrm{i}\varepsilon\sum_{k,m}kH_1^{km}(J)\exp\bigl[\mathrm{i}(k\alpha-m\nu t)\bigr]\,,\\ \dot\alpha&=\omega(J)+\varepsilon\sum_{k,m}\frac{\mathrm{d}H_1^{km}(J)}{\mathrm{d}J}\exp\bigl[\mathrm{i}(k\alpha-m\nu t)\bigr]\,,\end{aligned}\tag{18}$$

where $H_1^{-k,-m}=\overset{*}{H}{}_1^{k,m}$. It can be easily verified that if the condition

$$k\omega(J)-m\nu\approx0\tag{19}$$

is satisfied in Eqns (18), then a resonance arises.

To analyze the dynamics in the neighborhood of the resonance, we isolate the resonant term in decomposition (17) and consider the behavior of the system determined solely by this term. We fix the triplet of numbers k_0, m_0, J_0 such that the resonance condition is satisfied exactly,

$$k_0\omega(J_0)=m_0\nu\,,\tag{20}$$

and retain only the resonant harmonic in Eqns (18). Then,

$$\begin{aligned}\dot J&=-\mathrm{i}\varepsilon k_0H_1^{k_0m_0}(J)\exp\bigl[\mathrm{i}(k_0\alpha-m_0\nu t)\bigr]\\ &\quad+\mathrm{i}\varepsilon k_0\overset{*}{H}{}_1^{k_0m_0}(J)\exp\bigl[-\mathrm{i}(k_0\alpha-m_0\nu t)\bigr]\\ &=2\varepsilon k_0\,\frac{1}{2\mathrm{i}}\Bigl\{\bigl|H_1^{k_0m_0}(J)\bigr|\exp\bigl[\mathrm{i}(\psi+k_0\alpha-m_0\nu t)\bigr]\\ &\quad-\bigl|H_1^{k_0m_0}(J)\bigr|\exp\bigl[-\mathrm{i}(\psi+k_0\alpha-m_0\nu t)\bigr]\Bigr\}\\ &=2\varepsilon k_0\bigl|H_1^{k_0m_0}(J)\bigr|\sin(k_0\alpha-m_0\nu t+\psi)\,,\\ \dot\alpha&=\omega(J)+\varepsilon\,\frac{\mathrm{d}}{\mathrm{d}J}\,H_1^{k_0m_0}(J)\exp\bigl[\mathrm{i}(k_0\alpha-m_0\nu t)\bigr]\\ &\quad+\varepsilon\,\frac{\mathrm{d}}{\mathrm{d}J}\,\overset{*}{H}{}_1^{k_0m_0}(J)\exp\bigl[-\mathrm{i}(k_0\alpha-m_0\nu t)\bigr]\\ &=\omega(J)+2\varepsilon\,\frac{\mathrm{d}}{\mathrm{d}J}\bigl|H_1^{k_0m_0}(J)\bigr|\cos(k_0\alpha-m_0\nu t+\psi)\,.\end{aligned}$$

We introduce the notation $\varphi \equiv k_0\alpha - m_0 vt + \psi$ and $H_1^0 \equiv 2|H_1^{k_0 m_0}|$. Then Eqns (18) can be rewritten as

$$\dot{J} = \varepsilon k_0 H_1^0(J)\sin\varphi\,, \qquad \dot{\varphi} = k_0\omega(J) - m_0 v + \varepsilon k_0 \frac{\mathrm{d}H_1^0(J)}{\mathrm{d}J}\cos\varphi\,. \tag{21}$$

Retaining only the resonant term in expansion (17), we express Hamiltonian (16) as

$$H = H_0(J) + \varepsilon H_1^0 \cos\varphi\,. \tag{22}$$

We now assume that the quantity J is sufficiently close to J_0, such that the deviation $\Delta J \equiv J - J_0$ is small. In this case, $H_0(J)$ and $\omega(J)$ can be expanded into series in ΔJ:

$$H_0(J) = H_0(J_0) + \frac{\partial H_0}{\partial J}\Delta J + \frac{1}{2}\frac{\partial^2 H_0}{\partial J^2}(\Delta J)^2 + \ldots, \tag{23}$$

$$\omega(J) = \omega(J_0) + \frac{\partial \omega}{\partial J}\Delta J + \frac{1}{2}\frac{\partial^2 \omega}{\partial J^2}(\Delta J)^2 + \ldots\,. \tag{24}$$

We neglect the terms of orders higher than two in Eqn (23) and higher than one in Eqn (24). In addition, we take $H_1^0(J)$ at the point J_0, use equality (20), and neglect the term $\sim\varepsilon$ in the second of Eqns (21). Then system (21) and Hamiltonian (22) become

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta J = \varepsilon k_0 H_1^0 \sin\varphi\,, \qquad \dot{\varphi} = k_0 \frac{\mathrm{d}\omega(J_0)}{\mathrm{d}J}\Delta J\,, \tag{25}$$

$$H = H_0(J_0) + \omega(J_0)\Delta J + \frac{1}{2}\frac{\partial\omega(J_0)}{\partial J}(\Delta J)^2 + \varepsilon H_1^0\cos\varphi\,. \tag{26}$$

For convenience, we here use the notation $H_1^0 \equiv H_1^0(J_0)$. The variables ΔJ and φ are canonically conjugate for system (25), and system (25) itself is generated by the Hamiltonian

$$\bar{H} = \frac{1}{2}k_0\frac{\mathrm{d}\omega(J_0)}{\mathrm{d}J}(\Delta J)^2 + \varepsilon k_0 H_1^0\cos\varphi\,, \tag{27}$$

which is called the universal Hamiltonian of nonlinear resonance [59].

The canonical transformation that allows passing from the system with Hamiltonian (26) to the universal Hamiltonian of nonlinear resonance has the form

$$(\alpha, J) \to (k_0\alpha - m_0 vt,\ \Delta J)\,, \qquad v = \omega(J_0)\,, \qquad \bar{H} = k_0 H - m_0 v\Delta J - k_0 H_0(J_0)\,. \tag{28}$$

4.3 Width of the separatrix

We analyze expression (27) and Eqns (25). A similarity between $\bar{H}$ and the Hamiltonian of the nonlinear pendulum in (5) with $l = 1$ can immediately be noted. Indeed, the variable ΔJ plays the role of the momentum p and the quantity $m = (k_0\,\mathrm{d}\omega(J_0)/\mathrm{d}J)^{-1}$ has the meaning of the effective mass. Moreover, the substitution $\Omega_0^2 = \varepsilon k_0^2 H_1^0\left|\mathrm{d}\omega(J_0)/\mathrm{d}J\right|$ reduces system (25) to

$$\ddot{\varphi} - \Omega_0^2\sin\varphi = 0\,. \tag{29}$$

Equation (29) coincides with Eqn (6) up to a phase lag by π.

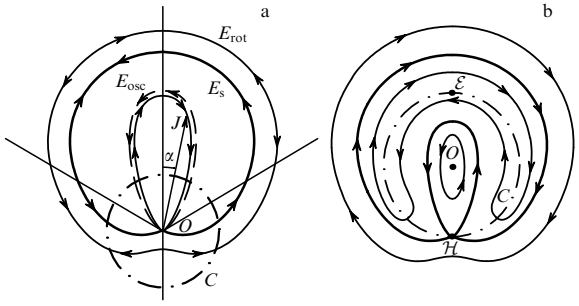

Figure 3. (a) First-order nonlinear resonance in the variables α, J and (b) in the variables specified by Eqn (28). Thin curves, phase oscillations; dot–dashed curve, unperturbed trajectory ($J = J_0$); thick curve, separatrix [60].

Let $k_0 = 1$, $m_0 = 1$ (a first-order resonance). The phase portrait in the original variables α, J is shown in Fig. 3a. The double drop-shaped curve, for a small energy $E = E_{\mathrm{osc}}$, corresponds to local oscillations of the system. In terms of the pendulum analogy, this corresponds to its oscillations. The outer curve corresponds to the energy $E = E_{\mathrm{rot}}$, which implies rotation in the case of the pendulum. The separatrix ($E = E_{\mathrm{s}}$) separates these two qualitatively different types of motion.

We now pass to the coordinate system specified by transformation (28). The resulting phase portrait is shown in Fig. 3b. We clarify this figure using the pendulum analogy. If the pendulum rotates oppositely to the rotation of the coordinate system, curves located around the point O inside the minor separatrix loop correspond to such motion. If the pendulum corotates with the coordinate system, curves located outside the major separatrix loop correspond to the motion of the pendulum. Horseshoe-shaped closed curves between the two separatrix loops correspond to oscillations of the pendulum. The oscillations occur around the elliptic point denoted as $\mathcal{E}$ in Fig. 3b. Both the major and minor separatrix loops pass through the hyperbolic point $\mathcal{H}$. The circle C represents the lower equilibrium of the pendulum. For the original system, the unperturbed trajectory for $J = J_0$ corresponds to the circle C. The directions of motion are shown by arrows in Fig. 3b.

The maximum distance between the two (major and minor) separatrix loops is known as the width of the nonlinear resonance (or the separatrix width). This quantity can easily be estimated. According to Eqn (27), we have the resonance width with respect to action

$$\max(\Delta J) = \sqrt{2\varepsilon H_1^0\left|\frac{\mathrm{d}\omega(J_0)}{\mathrm{d}J}\right|^{-1}} \tag{30}$$

and the resonance width with respect to frequency

$$\max(\Delta\omega) = \left|\frac{\mathrm{d}\omega(J_0)}{\mathrm{d}J}\right|\max(\Delta J) = \sqrt{2\varepsilon H_1^0\left|\frac{\mathrm{d}\omega(J_0)}{\mathrm{d}J}\right|}\,. \tag{31}$$

We now find how much the above assumptions are justified and determine the orders of magnitude of quantities (30) and (31). It follows from expression (22) that $H_0 \sim H_1^0$. But because $H_0 \sim J_0\omega(J_0)$,

$$H_0 \sim H_1^0 \sim J_0\omega(J_0)\,. \tag{32}$$

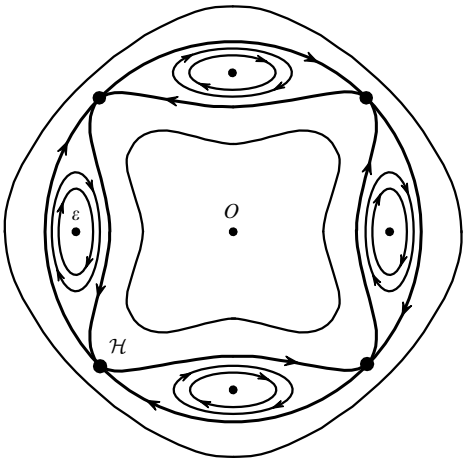

Figure 4. Nonlinear resonance at $k_0 = 4$, $m_0 = 1$.

Therefore, with (30) and (31), we obtain

$$\frac{\max(\Delta J)}{J_0} \sim \sqrt{\frac{\varepsilon}{\gamma}}\,, \qquad \frac{\max(\Delta\omega)}{\omega(J_0)} \sim \sqrt{\varepsilon\gamma}\,, \tag{33}$$

where $\gamma \equiv (\mathrm{d}\omega(J_0)/\mathrm{d}J)(J_0/\omega(J_0))$ is the so-called nonlinearity parameter. In deriving system (25), we neglected the term $\sim\varepsilon$ in Eqn (21) for $\dot{\varphi}$. This can be done [see Eqns (21) and (25)] if $(\mathrm{d}H_1^0(J_0)/\mathrm{d}J_0)\varepsilon \ll \left|\mathrm{d}\omega(J_0)/\mathrm{d}J\right|\Delta J$, or, in view of our estimate of the orders of magnitude, if $\varepsilon H_1^0/J_0 \ll \left|\mathrm{d}\omega(J_0)/\mathrm{d}J\right|\Delta J$. Therefore, in accordance with (32) and (33), we find

$$\gamma \gg \varepsilon\,. \tag{34}$$

In addition, we replaced $H_1^0(J)$ with $H_1^0(J_0)$. This substitution is equivalent to the condition $J - J_0 \ll 1$, or $\Delta J \ll J_0$. As can be easily seen from relations (33), the last inequality leads to inequality (34).

Retaining only the resonant term and neglecting the nonresonant terms is allowed if other frequencies in expansion (24) do not alter the dynamics qualitatively, i.e., if $\omega(J_0) \gg \left|\mathrm{d}\omega(J_0)/\mathrm{d}J\right|\Delta J$. Next, we use estimate (31) to obtain the condition $\gamma\varepsilon \ll 1$, which is satisfied for small γ and ε. With (34), we therefore have

$$\varepsilon \ll \gamma \ll \frac{1}{\varepsilon}\,. \tag{35}$$

Inequality (35) is called the condition of moderate nonlinearity [59, 60].

Thus, the above-presented approximate theory of nonlinear resonance is applicable if inequality (35) is satisfied. Therefore, there is no limit transition to the linear problem ($\gamma \to 0$).

Resonance condition (20) can generally be satisfied at any k_0. In this case, all assumptions and formulas remain unchanged. But because the phase is determined by the expression $\varphi = k_0\alpha - \nu t + \psi$, this leads to a garland of separatrix loops. The number of such loops is k_0. Therefore, the number of elliptic–hyperbolic pairs of points is also equal to k_0 (Fig. 4).

4.4 Internal resonances

The above description of nonlinear resonance can easily be generalized to systems with many degrees of freedom. For $n \geqslant 2$, however, resonances between the degrees of freedom of the system itself, i.e., internal resonances, are possible.

For clarity, we consider a system with two degrees of freedom ($n = 2$). The Hamiltonian of such a system can be written as

$$H = H_{01}(J_1) + H_{02}(J_2) + \varepsilon H_1(J_1, J_2, \alpha_1, \alpha_2)\,,$$

and the equations of motion are given by

$$\dot{J}_j = -\varepsilon\,\frac{\partial H_1}{\partial \alpha_j}\,, \qquad \dot{\alpha}_j = \frac{\partial}{\partial J_j}(H_{01} + H_{02}) + \varepsilon\,\frac{\partial H_1}{\partial J_j}\,, \qquad j = 1, 2\,.$$

According to relation (15), internal resonance occurs if the condition $k\omega_1(J_{01}) - m\omega_2(J_{02}) = 0$ is satisfied for certain integers k, m and certain values of the action variables J_{01}, J_{02}. We retain only the resonant harmonic in the decomposition

$$H_1 = \sum_{k,m} H_1^{km}(J_1, J_2)\exp\left[\mathrm{i}(k\alpha_1 - m\alpha_2)\right],$$

expand the functions H_{0j} and $\omega_j = \mathrm{d}H_{0j}/\mathrm{d}J_j$, $j = 1, 2$, in the neighborhood of the resonance (J_{01}, J_{02}), and use the above approximations to obtain the equations of motion

$$\begin{aligned} &\dot{J}_1 = \varepsilon k H_1^0 \sin\varphi\,, \qquad \dot{J}_2 = -\varepsilon m H_1^0 \sin\varphi\,,\\ &\dot{\varphi} = k\,\frac{\mathrm{d}\omega_1(J_{01})}{\mathrm{d}J_1}\,\Delta J_1 - m\,\frac{\mathrm{d}\omega_2(J_{02})}{\mathrm{d}J_2}\,\Delta J_2 \end{aligned} \tag{36}$$

and the universal Hamiltonian

$$\bar{H} = \frac{1}{2}\,\frac{\mathrm{d}\omega_1(J_{01})}{\mathrm{d}J_1}(\Delta J_1)^2 + \frac{1}{2}\,\frac{\mathrm{d}\omega_2(J_{02})}{\mathrm{d}J_2}(\Delta J_2)^2 + \varepsilon H_1^0\cos\varphi\,,$$

where we use the notation $\Delta J_j = J_j - J_{0j}$, $j = 1, 2$, $H_1^0\exp(\mathrm{i}\psi) \equiv 2H_1^{km}(J_{01}, J_{02}) = 2\left|H_1^{km}(J_{01}, J_{02})\right|\exp(\mathrm{i}\psi)$, and $\varphi = k\alpha_1 - m\alpha_2 + \psi$.

It can be easily seen that multiplying the first equation of system (36) by m and the second by k, and adding them, we obtain an additional integral of motion $mJ_1 + kJ_2 = \text{const}$. Therefore, Eqns (36) are integrable. But we can also proceed differently. Differentiation of the equation for $\dot{\varphi}$ with respect to time yields expression (29) with

$$\Omega_0^2 = \varepsilon H_1^0\left|k^2\,\frac{\mathrm{d}\omega_1(J_{01})}{\mathrm{d}J_1} + m^2\,\frac{\mathrm{d}\omega_2(J_{02})}{\mathrm{d}J_2}\right|.$$

If the quantities $\mathrm{d}\omega_1(J_{01})/\mathrm{d}J_1$ and $\mathrm{d}\omega_2(J_{02})/\mathrm{d}J_2$ have different signs, the expression inside the modulus vanishes, i.e., an additional degeneracy is possible in the system.

For systems with more than two degrees of freedom ($n \geqslant 3$), the number of additional integrals of motion is larger, but the dynamics are still governed by an equation similar to Eqn (29).

Thus, the ratio of frequencies plays an important role in the analysis of nearly integrable systems: the incommensurability of the frequencies typically determines a quasi-periodic trajectory that densely covers the torus. But a rational frequency ratio results in the emergence of resonances and modifies the structure of invariant surfaces.

4.5 Overlapping of resonances

Our analysis of nonlinear resonance was based on the assumption that the quantity J [see Eqn (20)] remains fixed. In other words, we assumed that only one isolated, primary nonlinear resonance exists. But it follows from (19) that the resonance relation can also be satisfied for other J_m, $m = 1, 2, \ldots$. In other words, there can be many primary resonances in the system, and Eqn (20) at $k_0 = 1$ must be rewritten as $\omega(J_m) = mv$. Therefore, we can introduce the distances between resonances: in action, $\delta J_m = J_{m+1} - J_m$, and in frequency, $\delta\omega_m = \omega(J_{m+1}) - \omega(J_m)$. If all J_m are sufficiently far from one another, i.e.,

$$\delta J_m \gg \Delta J_m \,, \tag{37}$$

where ΔJ_m is determined by Eqn (30), then the separatrices of the resonances do not intersect and no interactions between the resonances occur. If the initial conditions of the system are in the domain of one of the resonances, the dynamics can be described in the approximation where the effect of only this resonance is taken into account. If the initial point is in the domain between resonances, the system can be analyzed in the framework of the nonresonance approximation [54, 64].

Similarly, we can formulate the condition of the absence of frequency overlapping (interaction) of resonances:

$$\delta\omega_m \gg \Delta\omega_m \,. \tag{38}$$

We now assume that condition (37) or (38) is not satisfied and the resonances are close to one another, such that the corresponding separatrices can overlap. What happens in this case, known as the case of strong interaction between resonances?

We introduce the parameter K describing the degree of resonance overlapping:

$$K = \frac{\Delta\omega}{\delta\omega} \sim \frac{\Delta J}{\delta J} \,.$$

The overlapping parameter K was introduced by Chirikov to characterize the dynamics of Hamiltonian systems [58] (see also Refs [59, 65, 66]). If the interaction is weak, i.e., $K \ll 1$, the motion in the system should be regular and the phase trajectories should generally cover $(n - m)$-dimensional tori everywhere densely. But in the case of a strong overlapping, $K > 1$, the dynamics of the system turn out to be very complex and different from the dynamics in the periodic and quasi-periodic regimes.

The validity of the criterion of the onset of irregular motion based on the resonance-overlapping degree was many times confirmed numerically (see Refs [54, 59, 62]). In particular, Ref. [67] reported the origin of chaos with only two resonances overlapping.

A specific property of this criterion is that it can relatively simply be used to study particular systems. Indeed, it suffices to apply the above technique in the neighborhood of only one resonance in the approximation of the absence of all others. This makes the resonance-overlapping criterion very convenient in practice. At the same time, it is not universally applicable and in some cases requires more accurate formulations [54, 59, 66].

4.6 Higher-order resonances

Up to this point, we have considered only primary resonances. However, if the perturbation ε is sufficiently large, so-called secondary resonances, which modify or even completely destroy the integrals of the primary resonances, are possible. A secondary resonance is the resonance between the fundamental frequency of the unperturbed oscillation and the frequencies of harmonics of the primary-resonance phase oscillation. A specific feature of such a resonance is that garlands of separatrix loops appear near the elliptic point inside the primary resonance.

The small denominators of the secondary resonances can be removed as this was done for the primary resonance (see Ref. [54] for details). But such an approach has some specific features when applied to secondary resonances. In particular, the width of a secondary resonance depends on ε much more strongly than that of the primary resonance [$\sim \varepsilon^{1/2}$, see Eqns (30), (31)]. Therefore, secondary resonances are not important at small ε (in the case of small oscillations in the primary resonance). But if the perturbations are sufficiently strong, secondary resonances can affect the dynamics of the system as strongly as the primary resonances do.

In addition to secondary resonances, higher-order resonances are possible in systems; they can also substantially affect the motion. Thus, on the whole, the behavior of the system is highly involved and represents a hierarchy of very complex structures. We describe such a structure when we consider the dynamics near resonance separatrices. Our analysis clarifies some issues related to the origin and nature of dynamical chaos.

What is the structure of resonances near elliptic and hyperbolic points? This question is directly related to the nature of chaos.

5. Elements of the Kolmogorov – Arnol'd – Moser theory

A nonlinear resonance indicates that even small perturbations can substantially affect the dynamics of an integrable Hamiltonian system. Resonances modify the topology of phase trajectories and result in the formation of a chain of islands in phase space. The perturbation theory cannot describe such resonances, because regular solutions are strongly disturbed near them, which entails the emergence of small denominators and the divergence of series. This problem was already noted by Poincaré, who called it a fundamental problem of classical mechanics. However, it was solved only in the early 1960s, with the advent of the famous Kolmogorov – Arnol'd – Moser (KAM) theory [21 – 25].

To emphasize the physical significance of the KAM theory, we consider the problem of planetary motion around the Sun in accordance with the law of gravity. If the mutual influence of planets is neglected, we have a completely integrable system: planets move in ellipses according to Kepler's laws, and hence the dynamics of the system are in general quasi-periodic. If the interaction of the planets is taken into account, their orbits are deformed ellipses, which precess slowly. The precession is maximum for Mercury. The shift of its perihelion due to the influence of other planets [1] is estimated to be $\sim 532''$ per 100 years [68].

In principle, the interaction of planets should be taken into account using perturbative methods. But if resonances occur, such techniques yield diverging series and therefore do

[1] We emphasize that we consider only the Newtonian interaction.

not provide information on the dynamics of the system over long time intervals.

This planetary problem, reducible in the general case to the well-known N-body problem, is among the most important problems that have been raised in the course of the development of both mathematics and physics. In 1885, King Oscar II of Sweden and Norway, at the suggestion of G Mittag-Leffler, established a prize for solving this problem: "Given a system of arbitrarily many mass points that attract each other according to Newton's laws, try to find, under the assumption that no two points ever collide, a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly." (See Ref. [69] for a more detailed exposition of this and some other interesting points concerning the development of celestial mechanics and nonlinear dynamics.) Thus, the principal problem was not only to find a formal representation of solutions of the N-body problem in the form of series but also to prove their convergence. It is the latter issue that presented the main difficulty.

Without dwelling on the history of the question, we only note that the prize was awarded to Poincaré for his inestimable contribution to the solution of the N-body problem and related fundamental problems of dynamics.

Very much time was spent to realize that the problem posed is not simple and solving it requires employing novel powerful methods. Such methods were developed by the authors of the KAM theory, A N Kolmogorov, V I Arnol'd, and J Moser. The KAM theory not only permits constructing a converging expansion procedure but also, most importantly, gives a key to understanding the nature of the onset of chaos. In addition, this theory and its consequences proved to be very important in many areas of modern science, such as pure and applied mathematics, mechanics, physics, and even numerical analysis (!).

In recent years, substantial progress has been made in the KAM theory, and it remains a field of very active research (see Refs [53, 70, 71] and the references therein). This theory is presented in a large number of reviews and monographs (e.g., Refs [70, 72–76]). In Sections 5.1 and 5.2, we therefore describe only some qualitative statements of this theory and a number of its important consequences.

5.1 The Kolmogorov theorem

The famous Kolmogorov theorem (sometimes also called the KAM theorem) is at the heart of the KAM theory. It states that if a completely integrable system is perturbed sufficiently weakly, most nonresonant tori are preserved and only slightly deformed. In this case, the word 'most' means that all the resonant tori (corresponding to periodic motion) and part of nonresonant tori collapse, but this set is small compared to the set of nonresonant tori preserved under the perturbation.

The applicability conditions of the KAM theorem are as follows.

• The unperturbed Hamiltonian must satisfy the nondegeneracy condition

$$\det\left|\frac{\partial\omega_i}{\partial J_k}\right| = \det\left|\frac{\partial^2 H_0}{\partial J_i\,\partial J_k}\right| \neq 0\,, \qquad i = 1, 2, \ldots, n\,,$$

which means that the frequencies of the unperturbed system are functionally independent.

• The perturbation must be smooth, i.e., the Hamiltonian H_1 must have a sufficient number of derivatives.

• The system must be outside the neighborhood of a resonance, i.e.,

$$\left|\sum_j k_j\omega_j\right| > c|\mathbf{k}|^{-r}\,, \qquad \mathbf{k} = (k_1, k_2, \ldots, k_n)\,, \tag{39}$$

where r depends on the number of the degrees of freedom n and the constant c is determined by the magnitude of the perturbation εH_1 and the nonlinearity parameter γ. We note that the condition of moderate nonlinearity (35) can be obtained from inequality (39).

If the above conditions are satisfied, the meaning of the KAM theory is as follows. For most initial conditions, quasi-periodic dynamics are preserved in a nearly integrable system. However, there are initial conditions at which (mainly resonant) tori existing for $\varepsilon = 0$ break down and the motion becomes irregular. Precisely these tori break down under the perturbation and bring the system to chaos. This is why the theory of nonlinear resonance plays such an important role. The trajectories that originate in the region of destroyed tori can freely move in the energy space, which is manifested on the section surface as numerous randomly scattered points. Such trajectories turn out to be exponentially unstable with respect to small perturbations.

At small ε, interestingly, the tori remote from the resonance are preserved in the presence of arbitrary smooth perturbations. But the situation changes dramatically with increasing ε: the tori start breaking down and the domain of chaos starts expanding. Ultimately, this results in an overlapping of primary resonances and the onset of the phenomenon known in Hamiltonian mechanics as global chaos [54]. As global chaos occurs, no tori that were close to the tori in the unperturbed problem remain in the system, although some other tori can appear. The phase trajectory can move transversally to the chaotic layers in such a system.

However, at $\varepsilon \ll 1$, resonances do not overlap and the solutions lie on weakly deformed invariant tori. There is a qualitative difference in dynamics between systems with two degrees of freedom and those with more ($n > 2$) degrees of freedom.

5.2 The Arnol'd diffusion

In the case of two degrees of freedom, the phase space is four-dimensional, the energy hypersurface (or energy-level space) $H = E$ is three-dimensional, and the invariant tori are two-dimensional. Therefore, such tori can be represented as energy levels $H = E$ immersed in the three-dimensional space of the energy level $H = E$ (see Fig. 2). Therefore, the tori divide this space into disconnected domains. Thus, the destroyed tori turn out to be sandwiched between the tori preserved under the perturbation. The phase trajectory that originates at the location of such a destroyed torus (i.e., in the gap between two invariant tori) remains locked there forever. This means that the corresponding action variables are almost unchanged and remain near their initial values in the course of motion. Therefore, for systems with two degrees of freedom satisfying the conditions of Kolmogorov's theorem, for any initial conditions, no evolution occurs and global stability is preserved [72].

If $n > 2$, invariant tori no longer divide the $(2n-1)$-dimensional energy hypersurface into nonintersecting parts. In such systems, the dynamics are quasi-periodic for most

initial conditions. But there exist initial conditions at which the action variables recede slowly from their initial values. This can be easily understood from the fact that, at $n > 2$, the domains of destroyed tori merge, forming a cobweb. The phase point on the hypersurface of a given energy, moving along the threads of this cobweb, can approach any point of this hypersurface arbitrarily close. As investigations show, such an evolution of the action variables is random. This random walk over resonances around invariant tori is called the Arnol'd diffusion [54, 60, 70, 72].

Thus, small perturbations in systems with more than two degrees of freedom not only qualitatively modify the dynamics of the system but also can change the topology of phase trajectories, transforming them into a connected web.

A characteristic feature of the Arnol'd diffusion is its universality in the sense that no critical value of the perturbation ε necessary for the onset of such diffusion exists. In other words, diffusion always occurs, even at arbitrarily small ε. Obviously, the diffusion rate vanishes at progressively reduced perturbations. Therefore, for Hamiltonian systems with $n > 2$ degrees of freedom, the action variables can evolve slowly, which corresponds to lacking global stability. In the general case, this evolution is fairly slow and can be different in different parts of the phase space.

Another peculiarity of systems in which the Arnol'd diffusion is inherent consists in that their dynamics cannot exhibit a sudden transition to global chaos, which originates due to the overlapping of resonances. This is because regions with chaotic behavior in such systems are already unified into a web. In addition, the Arnol'd diffusion is very slow compared to the motion of trajectories in the region of global chaos.

The existence of diffusion trajectories was rigorously proved for the first time for a nonlinear Hamiltonian system of a special form [77]. There is no proof of the merging of chaotic trajectories into a web in the general case; however, fairly numerous examples are known in which this phenomenon is observed (see Refs [54, 59, 78–81]). The exact upper bound of the Arnol'd diffusion rate was obtained in Ref. [82] (see also Ref. [53] and the references therein). Various cases of diffusion, the accompanying physical phenomena, and estimates of the diffusion rate for various systems are discussed in monographs [53, 54].

We now consider the breakdown of the resonant tori and the origin of chaos in greater detail.

6. The nature of chaos

The issue of the nature of chaotic behavior traces back to the famous problem of intersection of separatrices in dynamical systems. This phenomenon was discovered long ago by Poincaré when he was studying the three-body problem [83]. But to understand the origin of chaos in systems of classical mechanics, the well-known Poincaré–Birkhoff fixed-point theorem [13, 84] should also be invoked (the mathematics of the issue can be surveyed using monograph [85]).

6.1 Twist map

We consider a system with two degrees of freedom, $n = 2$, and a Hamiltonian $H(q_1, p_1, q_2, p_2)$, although the results presented below can be generalized to the multidimensional case. In such a system, H is the total energy. Therefore, at a given value of $H = E$, the flow is always three-dimensional. This allows considering the corresponding Poincaré map instead of the continuous evolution of the system. Such a map can easily be constructed analytically.

For an integrable system in the angle–action variables, the phase space is a set of nested tori. We write the flow on one of them as

$$\alpha_1(t) = \omega_1 t + \alpha_1(0)\,, \qquad \alpha_2(t) = \omega_2 t + \alpha_2(0)\,,$$

where $\omega_1 = \omega_1(J_1, J_2) = \partial H/\partial J_1$ and $\omega_2 = \omega_2(J_1, J_2) = \partial H/\partial J_2$. Obviously, a complete revolution along the α_2 coordinate takes the time $t_2 = 2\pi/\omega_2$. By this time, the α_1 variable becomes

$$\alpha_1(t + t_2) = \alpha_1(t) + \omega_1 t_2 = \alpha_1(t) + \frac{2\pi\omega_1}{\omega_2} = \alpha_1(t) + 2\pi\rho(J_1)\,.$$

The quantity $\rho = \omega_1/\omega_2$ introduced here is called the rotation number. Because motion occurs in the energy-level space, $J_2 = J_2(J_1, E)$. Therefore, at a given E, the rotation number ρ is a function of J_1 only.

We now assume that the surface of a section is the plane (α_1, J_1), i.e., $\alpha_2 = \text{const}$. Then, for the intersection points x_k between the phase trajectory and this plane (Fig. 5a), we can write $x_k = \big(\alpha_1(t + kt_2), J_1\big)$. Thus, solutions of the original system on the given torus can be represented as the map P_0 that takes an intersection point x_k into the point x_{k+1} next in the invariant circle of 'radius' J_1. With the notation $J_k \equiv J_1(t + kt_2)$, it is possible to write such a displacement of points as

$$P_0 : \begin{cases} \alpha_{k+1} = \alpha_k + 2\pi\rho(J_k)\,, \\ J_{k+1} = J_k\,. \end{cases} \tag{40}$$

As can be seen from (40), the rotation number ρ depends on the radius of the circle in general.

If ρ is irrational, then the points x_k fill the entire circle as $k \to \infty$. If $\rho = l/m$ is rational, the points x_k map into one another successively, each m steps (Fig. 5b). Therefore, if this representation is used, we can speak of nonresonant and resonant circles.

As we pass from one circle to another, the rotation number changes. To be specific, we assume that $\rho(J)$ increases with increasing J. Then, as the origin of coordinates recedes, the angle through which the circle is rotated increases on average. This results in a twist of the radial line of points under the action of P_0. For this reason, transformation (40) is

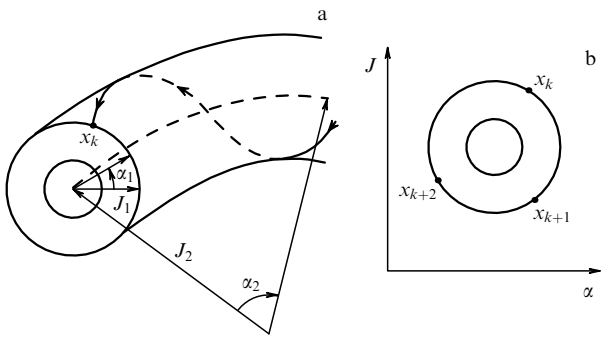

Figure 5. A system with two degrees of freedom (a) and its Poincaré map (b).

called the rotation map (or sometimes the twist map). In the general form, it can be written as

$$P_0(\mathcal{C}) = \mathcal{C}\,.$$

An important property of the map P_0 is that it is conservative.

We now consider perturbed system (9) with Hamiltonian (8). In the Poincaré section, taking the perturbation into account corresponds to adding new terms to the twist map:

$$P_\varepsilon : \begin{cases} \alpha_{k+1} = \alpha_k + 2\pi\rho(J_k) + \varepsilon f(J_k, \alpha_k)\,, \\ J_{k+1} = J_k + \varepsilon g(J_k, \alpha_k)\,, \end{cases} \tag{41}$$

where f and g are functions periodic in α. As before, the transformation P_ε must preserve the area; otherwise, invariant circles do not exist, as a rule. According to the KAM theory, under a small perturbation, $\varepsilon \ll 1$, most circles with irrational ρ are preserved, being only slightly deformed. We consider the behavior of the circles in the domain of rational, or resonant, values $\rho = l/m$, for which relation (39) is not valid. It is in this domain, as we already noted, chaotic motion originates.

6.2 Fixed-point theorem

We temporarily return to the integrable case. At $\rho(J) = l/m$, any point of the circle returns to its initial position after m steps, i.e., it is periodic with the period m. We let $\mathcal{C}$ denote this resonant circle and consider two nonresonant invariant circles $\mathcal{C}^+$ and $\mathcal{C}^-$ on both sides of $\mathcal{C}$ (Fig. 6a). Because $\rho(J)$ increases with J (see Section 6.1.), the irrational rotation numbers of such circles satisfy the respective inequalities $\rho > l/m$ and $\rho < l/m$. After applying the map P_0 m times (with the iterated map denoted by P_0^m), the points of the circle $\mathcal{C}^+$ are rotated through an angle larger than 2π and the points of the circle $\mathcal{C}^-$ through an angle smaller than 2π. It therefore appears that *relative to* $\mathcal{C}$, the map winds $\mathcal{C}^+$ counterclockwise and $\mathcal{C}^-$ clockwise (Fig. 6a).

We now consider perturbed map (41). According to the KAM theory, the circles $\mathcal{C}^+$ and $\mathcal{C}^-$ are preserved and only slightly deformed under the perturbation. Such closed circles, respectively denoted as C_ε^+ and $\mathcal{C}_\varepsilon^-$, are invariant under the transformation P_ε:

$$P_\varepsilon(\mathcal{C}_\varepsilon^+) = \mathcal{C}_\varepsilon^+\,, \qquad P_\varepsilon(\mathcal{C}_\varepsilon^-) = \mathcal{C}_\varepsilon^-\,.$$

We assume that the parameter ε is sufficiently small for the relative rotation of $\mathcal{C}_\varepsilon^+$ and $\mathcal{C}_\varepsilon^-$ to be preserved under the action of P_ε^m. Then, at any radius $\alpha = \text{const}$ between the curves $\mathcal{C}_\varepsilon^+$ and $\mathcal{C}_\varepsilon^-$, a point $J_p(\alpha, \varepsilon)$ can be found such that the map P_ε^m rotates its angular coordinates exactly through 2π. Therefore, the angular coordinates of this point are preserved under P_ε^m. At each radius issuing from the center, only one such point is located. Because the perturbation P_ε is smooth, these points $J_p(\alpha, \varepsilon)$ form a certain closed curve Γ_ε (Fig. 6b) that shrinks to $\mathcal{C}$ as $\varepsilon \to 0$. The curve Γ_ε is not invariant under P_ε. The action of P_ε^m amounts to radially shifting each point of Γ_ε. Thus, a new curve $P_\varepsilon^m(\Gamma_\varepsilon)$ is formed instead of Γ_ε (Fig. 7a).

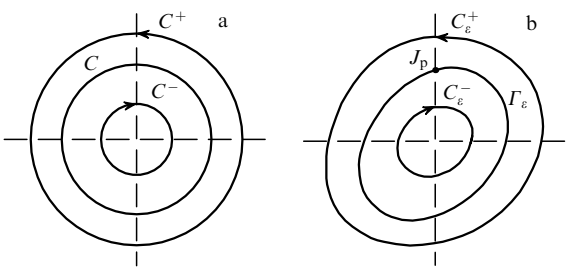

Figure 6. (a) Invariant circles $\mathcal{C}$, $\mathcal{C}^+$, and $\mathcal{C}^-$ of the unperturbed twist map P_0 for $\rho = l/m$, $\rho > l/m$, and $\rho < l/m$, respectively. (b) The result of the action of the perturbation P_ε.

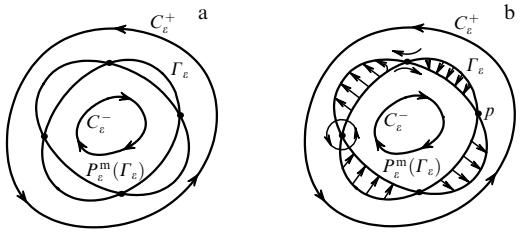

Figure 7. (a) Transformation of the curve Γ_ε into the curve $P_\varepsilon^m(\Gamma_\varepsilon)$ under the action of the map P_ε and (b) elliptic and hyperbolic points arising in this case.

Next, we recall that the map P_ε is conservative. Therefore, both curves Γ_ε and $P_\varepsilon^m(\Gamma_\varepsilon)$ must confine equal areas. Hence, the curve $P_\varepsilon^m(\Gamma_\varepsilon)$ can be neither inside nor outside the curve Γ_ε. Therefore, these curves must in general intersect[2] at an even number of points (Fig. 7a). In turn, each of these intersection points is fixed under the perturbed transformation P_ε^m.

This is the principal meaning of the Poincaré – Birkhoff fixed-point theorem [13, 84], according to which the perturbed twist map (41) with the rotation number $\rho = l/m$ has $2im$, $i = 1, 2, \ldots$, fixed points. Thus, as the resonant circle is perturbed (we recall that any point of this circle is fixed under the transformation P_0^m), only an even number $2im$ of fixed points is preserved.

We consider one of the points, p, of intersection between the curves Γ_ε and $P_\varepsilon^m(\Gamma_\varepsilon)$ (Fig. 7a). It is fixed under the map P_ε^m. The transformation P_ε acting on p generates a sequence of points $p, P_\varepsilon p, P_\varepsilon^2 p, \ldots, P_\varepsilon^{m-1} p$; after m iterations, the point p returns to its initial position. On the other hand, each of them is a fixed point for P_ε^m. Thus, there are m fixed points associated with the original point p. Because the curves Γ_ε and $P_\varepsilon^m(\Gamma_\varepsilon)$ intersect at an even number of points, we have a total of $2im$ fixed points.

6.3 Elliptic and hyperbolic points

A detailed consideration of the map P_ε in the neighborhood of various points p reveals the following qualitative difference. Points neighboring some points p remain near them and appear to rotate around them, while points located near other p tend to leave their neighborhood. Points of these two sorts alternate (Fig. 7b). Similar motion is observed in the phase space of a nonlinear pendulum (see Section 3.1). For this reason, such points are respectively called elliptic and hyperbolic. Elliptic points are surrounded by a family of closed trajectories that are invariant under P_ε^m and form 'islands'; hyperbolic points are connected by separatrices. This pattern is typical of weakly perturbed nonlinear systems, and it always emerges near a resonance.

Each island satisfies the KAM theory. Therefore, most nonresonant circles are preserved. But resonances are also

[2] We do not consider the exceptional case of tangency.

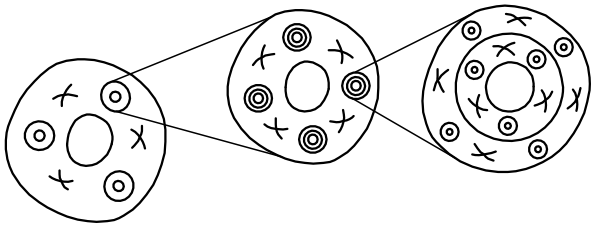

Figure 8. Destruction of tori with rational frequency ratios and the origin of a self-similar structure.

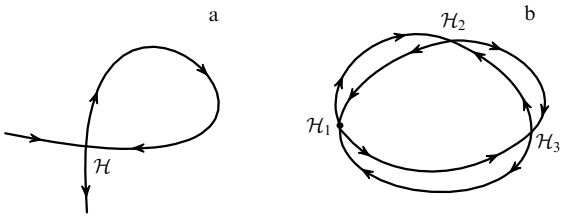

Figure 10. A homoclinic (a) and a heteroclinic (b) trajectory formed by separatrix branches.

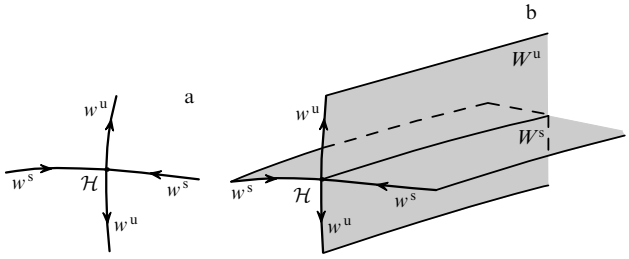

Figure 9. Stable (w^{s}) and unstable (w^{u}) directions of a hyperbolic point $\mathcal{H}$ in the Poincaré map (a) and the corresponding manifolds W^{s} and W^{u} in the phase space (b).

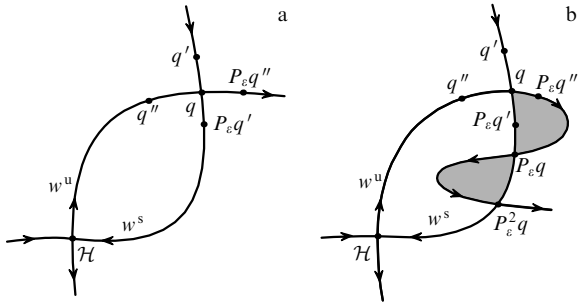

Figure 11. Map of neighboring points q' and q'' and the formation of loops from separatrices.

present here. According to the Poincaré – Birkhoff fixed-point theorem, a sequence of alternating elliptic and hyperbolic points appears in the neighborhood of each resonance, although on a smaller scale (Fig. 8). In turn, an island near each of these elliptic points reproduces the entire pattern in miniature. Closed curves in the neighborhood of elliptic points correspond to minor tori. According to the KAM theory, some of these minor tori are preserved; others collapse into smaller ones due to higher-order resonances, and so on, to infinity. Thus, the resonance results in a very complex pattern, which repeats itself on progressively smaller scales, thus being, in a certain sense, self-similar.

Following Refs [72, 86], we now consider the phenomena in the neighborhood of hyperbolic points. Each of these points is characterized by four invariant directions (or separatrix branches): two stable ones (w^{s}), which enter the hyperbolic point $\mathcal{H}$, and two unstable ones (w^{u}), which issue from $\mathcal{H}$ (Fig. 9). Because we are considering the Poincaré map, these curves correspond to surfaces, or invariant *manifolds*, either stable (W^{s}) or unstable (W^{u}), in the original phase space.

The oscillation period is infinite at the separatrix. Therefore, if some point q belongs to the stable branch w^{s}, it approaches $\mathcal{H}$ exponentially slowly under the action of the map P, i.e.,

$$\lim_{k\to\infty} P^k q \to \mathcal{H}\,.$$

But if $q \in w^{\mathrm{u}}$, it leaves the neighborhood of $\mathcal{H}$ exponentially slowly, i.e.,

$$\lim_{k\to\infty} P^{-k} q \to \mathcal{H}\,.$$

6.4 Splitting of separatrices. Homoclinic tangles

In the case of an integrable system, stable and unstable manifolds of hyperbolic points can be connected to one another, forming smooth structures. In a Poincaré map, such a structure appears as a smooth transition from an unstable to a stable separatrix branch. Separatrix branches can either close to the same hyperbolic point (Fig. 10a) or connect several such points, forming garlands (Fig. 10b). In the first case, a stable (w^{s}) and an unstable (w^{u}) separatrix branch forms a loop. This loop, known as a homoclinic loop, is a doubly asymptotic trajectory with the property that any point q on such a loop always approaches $\mathcal{H}$, i.e.,

$$\lim_{k\to\pm\infty} P^k q \to \mathcal{H}\,.$$

An elliptic point exists inside the homoclinic curve.

In the second case, the curves consisting of stable and unstable separatrix branches are called heteroclinic trajectories. The motion along such trajectories implies exponentially slow recession from one hyperbolic point and approach to another.

If a perturbation is present, the separatrix branches no longer form smooth homoclinic and heteroclinic junctions but can intersect. The point of the intersection of a stable and an unstable separatrix branches of the same resonance is called a homoclinic point. If a stable and an unstable branch of different hyperbolic points (resonances) intersect, a heteroclinic point appears. We now consider how such points evolve under the action of the map P_ε.

Let q be a homoclinic point and $q' \in w^{\mathrm{s}}$ and $q'' \in w^{\mathrm{u}}$ be its neighboring points (Fig. 11a). Under the action of P_ε, the points q' and q'' must be mapped into the points $P_\varepsilon q'$ and $P_\varepsilon q''$. Where should we find the point $P_\varepsilon q$ into which q is mapped by P_ε? As can be seen from Fig. 11a, taking the directions of motion in the stable and unstable branches into account, q is located before the points q' and q''. Therefore, because P_ε is continuous, this point must be taken into a point that is also located before the points $P_\varepsilon q'$ and $P_\varepsilon q''$. This means that a new intersection, i.e., a new homoclinic point $P_\varepsilon q$, must arise (Fig. 11b).

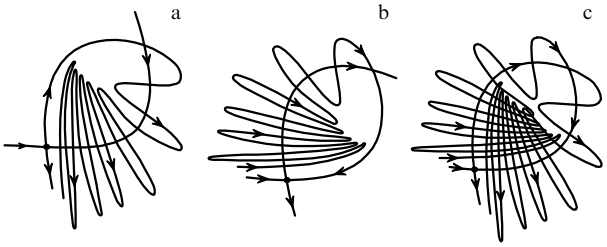

Figure 12. Homoclinic tangles in the neighborhood of a hyperbolic point.

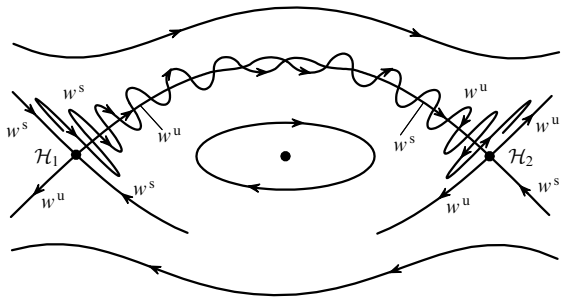

Figure 13. Formation of heteroclinic tangles.

In other words, because the branches w^s and w^u are invariant, we should remain at both separatrix branches under the action of the map. Therefore, a new intersection point $P_\varepsilon q$ emerges, and a loop forms between the points q and $P_\varepsilon q$.

By similar considerations, we can easily find that the point $P_\varepsilon q$ is mapped into the point $P_\varepsilon^2 q$, forming another loop (Fig. 11b). Because the new point $P_\varepsilon^2 q$ is closer to the hyperbolic point $\mathcal{H}$, the distance between $P_\varepsilon^2 q$ and $P_\varepsilon q$ is shorter than the distance between $P_\varepsilon q$ and q. We recall that the conservation of phase volume is a property of the map P_ε. This means that the areas confined by the loops between q and $P_\varepsilon q$ and between $P_\varepsilon q$ and $P_\varepsilon^2 q$ must be equal. Therefore, the second loop is more stretched and curved than the first one.

Continuing our considerations leads us to the conclusion that an infinite number of intersections of separatrix branches eventually arise, which become progressively closer, and the loops themselves become longer and thinner (Fig. 12a). The stable separatrix branch w^s behaves similarly (Fig. 12b). This can easily be understood if we move along w^s in the opposite direction. Thus, the pattern in the neighborhood of hyperbolic points is on the whole extremely complex (Fig. 12c).

For heteroclinic trajectories, the intersection of separatrices results in the formation of somewhat different structures. The unstable branch w^u issuing from the hyperbolic point $\mathcal{H}_1$ oscillates in approaching a hyperbolic point $\mathcal{H}_2$ located on the right. In contrast, the stable point w^s oscillates in receding from $\mathcal{H}_1$ (Fig. 13). The stable and unstable branches located in the bottom part of the figure (not shown in full) behave similarly.

Thus, due to a perturbation introduced into the integrable system, the separatrices are no longer smooth, as in Fig. 10, but split in an intricate manner. This phenomenon is termed the splitting of separatrices. The structures formed by separatrix branches in the domain of hyperbolic points are called homoclinic and heteroclinic tangles. It is such complex behavior of separatrices that gives rise to chaos in determi-

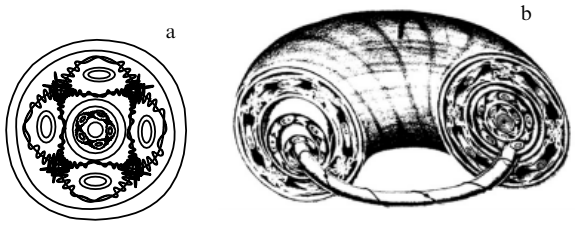

Figure 14. Structure of the phase space of perturbed Hamiltonian systems with two degrees of freedom: (a) in the Poincaré map and (b) in the phase space [87].

nistic systems. Invariant tori cannot exist in the domain of homoclinic tangles. There, the systems are not integrable, exhibit the property of the exponential instability of trajectories under small perturbations, and therefore behave chaotically. However, at some different initial conditions from the neighborhood of the surviving tori, the dynamics of the system are regular.

We can now imagine the entire pattern arising in phase space and in the Poincaré section of the set of invariant tori. The destruction of resonant circles under perturbations is accompanied by the occurrence of $2im$ hyperbolic and elliptic points. In the neighborhood of each elliptic point, a family of closed invariant curves is present, and some of them are also destroyed by the perturbation. This leads to a smaller chain of elliptic and hyperbolic points. In the neighborhood of each hyperbolic point, the separatrices split, their tangles appear, etc. (Fig. 14a).

However, all this occurs not in the Poincaré section but in the phase space formed by a set of tori. Thus, the general pattern of motion of the phase trajectories turns out to be highly involved (Fig. 14b). This structure repeats itself on progressively smaller scales and is typical of nearly integrable systems.

As the perturbation increases, some nonresonant circles also break down. But if the perturbation H_1 is small, chaotic trajectories exist only in the phase-space domain bounded by invariant curves.

An elementary exposition of the foundations of the theory, the questions concerning the physical aspects of separatrix intersection, and a historical background are given in monograph [69]. A rigorous presentation of related topics is made in Refs [53, 72, 86, 88]. The fixed-point theorem, its qualitative description, and some applications can be found in book [13]. Mathematical approaches to the problem of separatrix intersection are presented in Refs [32, 63, 71, 88–90].

7. The Mel'nikov method

The distance between separatrices can be estimated analytically and the conditions for the onset of homoclinic and heteroclinic chaos can be obtained using the Mel'nikov's theory [91] (see also Refs [54, 63, 88, 89]).

To simplify the presentation, we consider a two-dimensional autonomous system with a periodic external perturbation

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon \mathbf{f}_1(\mathbf{x}, t)\,, \tag{42}$$

[where $\mathbf{x} = (x_1, x_2)$, $\mathbf{f}_0 = (f_{01}, f_{02})$, $\mathbf{f}_1 = (f_{11}, f_{12})$, and $\mathbf{f}_1(\mathbf{x}, t) = \mathbf{f}_1(\mathbf{x}, t + T)$] that has a unique hyperbolic

point $\mathcal{H}_0$. We assume that the unperturbed system ($\varepsilon = 0$) has a homoclinic loop $w_0(t)$ (Fig. 10a),

$$\lim_{t\to\pm\infty} w_0(t) = \mathcal{H}_0 .$$

The perturbation results in splitting this loop, such that the incoming (w_0^{s}) and outgoing (w_0^{u}) branches no longer form a single curve.

To find the condition of the presence of chaos, it is necessary to perturbatively calculate the distance $D(t, t_0)$ between the branches of the separatrix at some instant t_0. Then, if $D(t, t_0)$ does not change its sign, the branches w_0^{s} and w_0^{u} do not intersect. But if the sign of $D(t, t_0)$ changes at some t_0, homoclinic tangles appear (Fig. 12c) and chaotic motion sets in. This is considered formally in Section 7.1.

7.1 The Mel'nikov function

The method proposed in Ref. [91] is based on a comparison of first-order terms in the expansion of separatrix solutions into series in the perturbation parameter ε. In other words, to calculate $D(t, t_0)$, it is sufficient to find a stable branch w^{s} and an unstable branch w^{u} in the first approximation.

Let $w^{\mathrm{s,u}}(t, t_0) = w_0(t - t_0) + \varepsilon w_1^{\mathrm{s,u}}(t, t_0)$. Then, using Eqn (42), we obtain, in the first order in ε,

$$\frac{\mathrm{d}w_1^{\mathrm{s,u}}}{\mathrm{d}t} = \hat{M}(w_0) w_1^{\mathrm{s,u}} + \varepsilon \mathbf{f}_1\big(w_0(t - t_0), t\big) , \tag{43}$$

where $\hat{M}(w_0)$ is the Jacobi matrix

$$\hat{M}(w_0) = \begin{pmatrix} \dfrac{\partial f_{01}}{\partial x_1} & \dfrac{\partial f_{01}}{\partial x_2} \\ \dfrac{\partial f_{02}}{\partial x_1} & \dfrac{\partial f_{02}}{\partial x_2} \end{pmatrix}$$

taken at the unperturbed homoclinic trajectory $w_0(t - t_0)$.

If a perturbation is introduced into an integrable system, the original hyperbolic point $\mathcal{H}_0$ displaces to a point $\mathcal{H}_{\mathrm{p}}$. Then, obviously, the stable and unstable branches satisfy the relation

$$\lim_{t\to\infty} w^{\mathrm{s}} = \lim_{t\to-\infty} w^{\mathrm{u}} = \mathcal{H}_{\mathrm{p}} .$$

We introduce the vector

$$\mathbf{d}(t, t_0) = w^{\mathrm{s}}(t, t_0) - w^{\mathrm{u}}(t, t_0) .$$

In the first approximation, $\mathbf{d}(t, t_0) = w_1^{\mathrm{s}}(t, t_0) - w_1^{\mathrm{u}}(t, t_0)$. According to Ref. [91], the distance $D(t, t_0)$ between the stable and unstable branches of the separatrix is defined as the projection of $\mathbf{d}$ onto the normal $\mathbf{N}$ to the unperturbed loop w_0 at the instant t (Fig. 15):

$$D(t, t_0) = \mathbf{N}\mathbf{d} . \tag{44}$$

At $\varepsilon = 0$, using Eqn (42), we can determine the vector $\mathbf{N}$ to be

$$\mathbf{N}(t, t_0) = \begin{pmatrix} -f_{02}(w_0) \\ f_{01}(w_0) \end{pmatrix} .$$

If we introduce the operator $\wedge$ by the relation $\mathbf{x} \wedge \mathbf{y} = x_1 y_2 - x_2 y_1$, Eqn (44) can be represented in a simpler form as $D(t, t_0) = \mathbf{f}_0 \wedge \mathbf{d}$. We rewrite this expression as

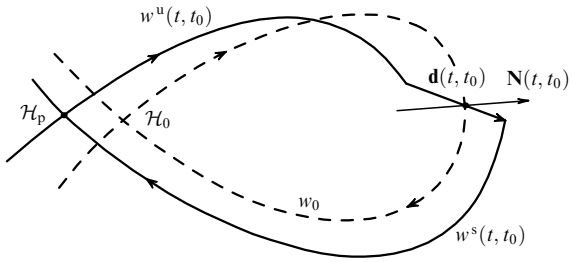

Figure 15. Determining the distance between the separatrix branches, $D(t, t_0) = \mathbf{N}\,\mathbf{d}$ [54].

$$D = D^{\mathrm{s}} - D^{\mathrm{u}} , \tag{45}$$

where $D^{\mathrm{s,u}}(t, t_0) = \mathbf{f}_0 \wedge w_1^{\mathrm{s,u}}$. Now, differentiating D^{s} with respect to time yields $\dot{D}^{\mathrm{s}} = \dot{\mathbf{f}}_0 \wedge w_1^{\mathrm{s}} + \mathbf{f}_0 \wedge \dot{w}_1^{\mathrm{s}} = \big(\hat{M}(w_0)\,\dot{w}_0\big) \wedge w_1^{\mathrm{s}} + \mathbf{f}_0 \wedge \dot{w}_1^{\mathrm{s}}$. Therefore, because $\dot{w}_0 = \mathbf{f}_0$, it follows from (43) that

$$\dot{D}^{\mathrm{s}} = \dot{\mathbf{f}}_0 \wedge w_1^{\mathrm{s}} + \mathbf{f}_0 \wedge \dot{w}_1^{\mathrm{s}} = \big(\hat{M}(w_0)\,\mathbf{f}_0\big) \wedge w_1^{\mathrm{s}} + \mathbf{f}_0 \wedge \big(\hat{M}(w_0)\,w_1^{\mathrm{s}}\big) + \mathbf{f}_0 \wedge \mathbf{f}_1$$

or

$$\dot{D}^{\mathrm{s}} = \mathrm{Sp}\,\hat{M}(w_0)\mathbf{f}_0 \wedge w_1^{\mathrm{s}} + \mathbf{f}_0 \wedge \mathbf{f}_1 = \mathrm{Sp}\,\hat{M}(w_0) D^{\mathrm{s}} + \mathbf{f}_0 \wedge \mathbf{f}_1 . \tag{46}$$

Asymptotically, we have the distance

$$D^{\mathrm{s}}(t, t_0)\big|_{t\to\infty} = \mathbf{f}_0(\mathcal{H}_0) \wedge w_1^{\mathrm{s}} = 0 .$$

If the unperturbed system is weakly dissipative, $\mathrm{Sp}\,\hat{M} \approx 0$. Therefore, integrating Eqn (46) over the semi-interval $[t_0, \infty)$, we obtain

$$D^{\mathrm{s}}(t_0, t_0) = -\int_{t_0}^{+\infty} \mathbf{f}_0 \wedge \mathbf{f}_1\,\mathrm{d}t .$$

Similarly, for D^{u} we find that

$$D^{\mathrm{u}}(t_0, t_0) = \int_{-\infty}^{t_0} \mathbf{f}_0 \wedge \mathbf{f}_1\,\mathrm{d}t .$$

Because the sought function D is determined by Eqn (45), we finally obtain

$$D = -\int_{-\infty}^{\infty} \mathbf{f}_0 \wedge \mathbf{f}_1\,\mathrm{d}t . \tag{47}$$

The obtained function (sometimes called the Mel'nikov function) describes the splitting of separatrices of a hyperbolic point due to the introduction of a small perturbation into an integrable system. If function (47) is sign-alternating, the stable and the unstable branches intersect forming homoclinic tangles, which results in the onset of chaotic dynamics.

Despite its relative simplicity, the Mel'nikov method has proven itself in applications (in particular, to weakly dissipative systems). Moreover, for certain problems, it is possible to calculate integral (47) analytically and compute the threshold of the onset of chaos.

7.2 The Duffing oscillator and nonlinear pendulum

As the first example of using the above analytic approach and estimating its accuracy, we consider two models that are used

to test various applications of the methods of analytic treatment of nonlinear systems: the Duffing oscillator [92] and a nonlinear pendulum with dissipation and periodic excitation.

The Duffing oscillator is a system with two stable equilibrium states divided by a single separatrix and with excitation. The equation describing such a system is called the Duffing equation:

$$\ddot{x} + \delta\dot{x} - x + x^3 = \gamma\cos\omega t\,. \tag{48}$$

Equation (48) has been comprehensively studied, on the one hand, numerically and analytically and, on the other hand, experimentally (see, e.g., Refs [88, 93–95] and the references therein).

At $\delta = 0$, Eqn (48) always has chaotic solutions near the separatrix, which are bounded by invariant curves. In the dissipative case, no such curves exist, and hence, with the passage of time, a chaotic trajectory finds its way to the attraction region of a stable focus or limit cycle. In this case, transient chaos is observed in the system.

The Mel'nikov function for Eqn (48) can be estimated without difficulty (see Ref. [96]). We rewrite Eqn (46) in a different form, $\dot{x} = y$, $\dot{y} = x - x^3 - \delta y + \gamma\cos\omega t$. The unperturbed Hamiltonian is then given by $H_0 = y^2/2 - x^2/2 + x^4/4$. The equations of motion at the separatrix can be written as [96]

$$x_0(t) = \frac{\sqrt{2}}{\cosh t}\,, \qquad y_0(t) = -\sqrt{2}\,\frac{\sinh t}{\cosh^2 t}\,.$$

Using these relations, we next find $\mathbf{f}_0 \wedge \mathbf{f}_1 = y_0[\gamma\cos\omega t - \delta y_0]$. The integral of this function can be analytically calculated as

$$D(t_0) = \sqrt{2}\,\pi\gamma\omega\,\frac{\sin\omega t_0}{\cosh(\pi\omega/2)} + \frac{4}{3}\,\delta\,.$$

Hence, the condition of intersection of separatrices can be written as

$$\delta < \delta_c = \frac{3\sqrt{2}\,\pi\gamma\omega}{4\cosh(\pi\omega/2)}\,. \tag{49}$$

A generalized Duffing equation is also often considered; it describes an oscillator with a parametrically perturbed cubic term,

$$\ddot{x} + \delta\dot{x} - x + \beta(1 + \eta\cos\Omega t)x^3 = \gamma\cos\omega t\,, \tag{50}$$

where $\eta \ll 1$ is the amplitude and Ω is the frequency of the parametric perturbation. This equation is interesting because it has two excitation frequencies, ω and Ω.

The Mel'nikov function for Eqn (50) can be written as (see Ref. [97])

$$D(t_0) = -A\sin\omega t_0 + B\sin\Omega t_0 + C\,,$$

where

$$A = \pi\kappa\gamma\omega\,\operatorname{sech}\frac{\pi\omega}{2}\,,$$

$$B = \frac{1}{24}\,\pi\beta\eta\kappa^4\Omega^2(4 + \Omega^2)\operatorname{cosech}\frac{\pi\Omega}{2}\,,$$

$$C = \frac{2}{3}\,\kappa^2\delta\,, \qquad \kappa = \sqrt{\frac{2}{\beta}}\,.$$

If $\omega = \Omega$, the values of δ at which D can change sign can be explicitly written as

$$\delta < \delta_c \equiv \frac{3}{2\kappa^2}\,|B - A|\,. \tag{51}$$

For $\omega \neq \Omega$, we must additionally consider the cases of commensurate and incommensurate frequencies ω and Ω. In the former case,

$$\delta < \delta_c \equiv \frac{3(A + B)}{2\kappa^2}\,.$$

For incommensurate frequencies (see Ref. [98]),

$$\frac{3}{2\kappa^2}\,|A - B| < \delta_c < \frac{3}{2\kappa^2}(A + B)\,.$$

For a nonlinear pendulum that obeys the equation $\ddot{x} + \alpha\dot{x} + \sin x = \gamma\cos\omega t$, the Mel'nikov function can also be calculated analytically [56, 57],

$$D(t_0) = -4\alpha B\left(\frac{1}{2}\,, 1\right) \pm \frac{2\pi\gamma\cos\omega t_0}{\cosh\pi\omega/2}\,,$$

where $B(r, s)$ is the Euler β function. Based on this, the threshold of the onset of chaotic motion can also be determined.

As found in Refs [56, 96, 98], the obtained analytic results agree very well with numerical estimates based on calculations of the maximal Lyapunov exponent. For example, at $\beta = 8$, $\gamma = 0.114$, $\eta = 0.03$, and $\omega = \Omega = 1.1$ [see Eqn (50)], we obtain $\delta_c = 0.3798$ from (51). A numerical analysis yields $\delta_c = 0.378$ as the threshold of the onset of chaos.

Thus, in some cases, the Mel'nikov method allows analytically determining the threshold of the onset of chaos, in particular, for multidimensional systems [99]. We emphasize, however, that this is a local criterion that can be used in the neighborhood of unperturbed separatrix loops.

8. Principal properties of chaotic systems

We now describe important properties inherent in chaotic systems: unpredictability, irreversibility, and decay of correlations. However, we start by introducing the concept of mixing.

8.1 Ergodicity and mixing

We assume that the trajectories of a dynamical system are localized in a certain bounded domain D of the phase space. The motion of the dynamical system is said to be ergodic (or the dynamical system is called ergodic) if, for an arbitrary integrable function $h(\mathbf{x})$, at almost all initial conditions $\mathbf{x}_0$, the temporal and phase averages of $h(\mathbf{x})$ coincide:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T h(\mathbf{F}^t\mathbf{x}_0)\,\mathrm{d}t = \int_{(D)} h(\mathbf{x})\,\mathrm{d}\mu\,,$$
$$\mu(D) = \int_{(D)}\mathrm{d}\mu = 1\,. \tag{52}$$

Thus, if the motion of a conservative system is ergodic, a phase trajectory, with the passage of time, uniformly and densely covers the hypersurface specified by all integrals of motion in the n-dimensional phase space. A characteristic property of the ergodic motion of a Hamiltonian system is the invariance of the shape of a small domain. As time passes,

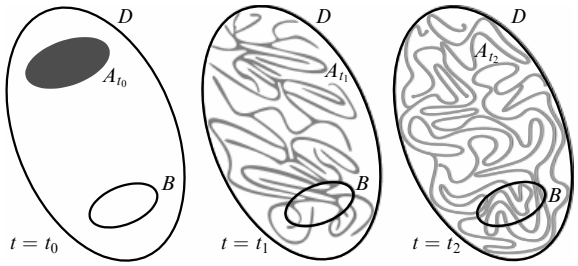

Figure 16. Evolution of a small domain during mixing.

such a domain simply moves over the torus intersecting each of its segments infinitely many times.

It is obvious, however, that 'ergodic' does not necessarily mean 'chaotic.' In addition, the motion of a chaotic system must also be locally unstable, which corresponds to the presence of homoclinic and heteroclinic tangles. Because of such an instability, an initial elementary volume, moving over the entire energetically accessible hypersurface (as for ergodic flows), undergoes strong deformations with time. Such systems are called systems with mixing.

The concept of mixing is very simple in essence. For illustration, we consider Gibbs's classic example. We assume that a vessel contains 30% ink and 70% water and these fluids are not mixed at the initial instant. We then thoroughly shake the contents of the vessel. It is natural to expect that some time later (after the mixing time has elapsed), any portion of the obtained mixture will consist of 30% ink and 70% water. Formalization of such a process leads to the concept of mixing.

We consider two arbitrary domains A and B with measures $\mu(A)$ and $\mu(B)$ inside the domain $D \subset M$ and assume that the domain B remains motionless, while the domain A evolves under the action of the transformation $\mathbf{F}^t$: $A_t = \mathbf{F}^t A$ (Fig. 16). Let $A_t \bigcap B$ be the collection of all parts of A_t that were at inside the motionless domain B at an instant t. Dynamical system (1) is called a system with mixing (and, accordingly, the phase flow $\mathbf{F}^t$ is said to be mixing) if there exists the limit

$$\lim_{t\to\infty} \frac{\mu\left(A_t \bigcap B\right)}{\mu(B)} = \mu(A)\,. \tag{53}$$

We clarify the meaning of this definition. First, we recall that because the considered motion is conservative, the measure of the domain A is preserved in the course of motion, i.e., $\mu(A_t) = \mu(A)$. Moreover, $\mu(A) = \mu(A_t)/\mu(D)$ is the relative volume occupied by the domain A in D. At the same time, the ratio $\mu\left(A_t \bigcap B\right)/\mu(B)$ is the relative volume of the domain B occupied by the pieces of A_t that found their way there by time t. It follows from definition (53) that these two ratios coincide as $t \to \infty$, irrespective of the sizes, shapes, and mutual positions of the domains A and B.

A mixing flow arises if phase-space points that are close to one another initially move in exponentially diverging trajectories. This fact was noted long ago by E Hopf as he analyzed motion on a negative-curvature manifold [100]. It can be easily understood that mixing is inherent in dynamical systems in the domain of homoclinic and heteroclinic tangles.

Mixing implies ergodicity. But the converse is not true: ergodicity does not imply mixing. Therefore, mixing is a much stronger property than ergodicity. If mixing occurs in a system, its behavior can naturally be considered chaotic. Moreover, the property of mixing has three very important consequences observed in natural systems — unpredictability, irreversibility, and decay of temporal correlations.

8.2 Unpredictability and irreversibility

The unpredictability and irreversibility of the temporal evolution of systems described by reversible dynamic equations (e.g., Hamiltonian equations) can be described based on the concepts of mixing and openness.

As is known, because of unavoidable fluctuations and external perturbations, the state of any physical system can be known only to a certain finite accuracy. This can be interpreted as the distribution of initial conditions $\mathbf{x}_0$ in some domain $\Omega_\varepsilon \subset D$ with a characteristic size ε. Then, if the system has the property of mixing, such an initial distribution spreads over the entire domain D as $t \to \infty$. In other words, if the position of a phase point is initially known to a finite accuracy (i.e., we only knew at $t = 0$ that this points belonged to a certain region Ω_ε), it is impossible to predict the position of this point after a sufficiently long lapse of time. Thus, unpredictability originates in systems with mixing. The sensitivity of the system to initial conditions (their small perturbations) plays a crucial role here.

If an isolated Hamiltonian system is ergodic, then it passes through all energetically possible states in the course of time and its phase trajectory visits all accessible domains of the phase space. The time of residence in a given domain is then proportional to the volume of this domain. As $t \to \infty$, any of these states is observable. If the number of the degrees of freedom of the system is sufficiently large, then the time in which the system reaches a certain particular state (assumed to be the initial one) can be very long. This obviously implies the irreversibility of closed systems with very many degrees of freedom. This explanation of irreversibility is now generally accepted (see also Refs [101, 103] in this context).

For low-dimensional open systems, external fluctuations play an important role in the emergence of irreversibility. If mixing occurs in the system, then, inside any accessible part of a domain D in the phase space M, parts of various initial domains $A \subset D$ find themselves after the lapse of a sufficiently long time (see Fig. 16). Therefore, if we only know that the particle is located at a finite time within a small domain with a size of the order of ε, we cannot indicate where it was at the initial time. In other words, the openness and, therefore, the unavoidable inaccuracy in determining the current state of the system play an important role in this case.

It can be easily seen that there is no contradiction with the existence of reversible equations of motion that describe motion with mixing (i.e., random motion, in a sense). The point is that irreversibility is related to coarsening or rounding of some quantities. In this case, part of the information contained in the exact solution is lost from the very beginning. The state of such a system can only become progressively less definite with its motion. For this reason, describing chaotic systems immediately necessitates using a coarsened analysis.

Thus, the concept of mixing corresponds to our intuitive idea of systems with complex and irregular regimes of motion. But not all chaotic motion must be mixing. Definition (53) imposes a number of strict limitations that are not always satisfied in reality.

8.3 Decay of correlations

A consequence of mixing in a system is a very important property that is frequently used as the criterion of chaotic behavior: the decay of temporal correlations.

We consider arbitrary integrable functions f and g in a certain domain D of the phase space M of a dynamical system. We write their phase averages as

$$\langle f \rangle = \frac{1}{V_D} \int_{(D)} f(\mathbf{x})\, \mathrm{d}x_1 \ldots \mathrm{d}x_n \,,$$

$$\langle g \rangle = \frac{1}{V_D} \int_{(D)} g(\mathbf{x})\, \mathrm{d}x_1 \ldots \mathrm{d}x_n \,.$$

Let the initial state of the system be specified by the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. Then, after the lapse of some time, the system comes to the state $\mathbf{x}(t)$ that depends on $\mathbf{x}(0)$ in view of the uniqueness of the solution, $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$. We define the correlator of the functions f and g as

$$\big\langle f\big(\mathbf{x}(\mathbf{x}_0, t)\big)\, g(\mathbf{x}_0)\big\rangle = \frac{1}{V_D} \int_{(D)} f\big(\mathbf{x}(\mathbf{x}_0, t)\big)\, g(\mathbf{x}_0)\, \mathrm{d}\mathbf{x}_0 \,.$$

This can be rewritten in the form

$$\big\langle f(\mathbf{x})\, g\big(\mathbf{x}_0(\mathbf{x}, t)\big)\big\rangle = \frac{1}{V_D} \int_{(D)} f(\mathbf{x})\, g\big(\mathbf{x}_0(\mathbf{x}, t)\big)\, \mathrm{d}\mathbf{x} \,.$$

It can be easily shown (see, e.g., Ref. [103]) that if mixing occurs in the system, then we have

$$\big\langle f(\mathbf{x})\, g(\mathbf{x}_0)\big\rangle = \langle f \rangle \langle g \rangle \tag{54}$$

for arbitrary functions f and g as $t \to \infty$. Property (54) is called the decay of temporal correlations. In other words, a system with mixing 'forgets' its initial conditions with the passage of time.

The decay of correlations is among the very important consequences of chaotic behavior and consists of the fact that after the lapse of a time interval equal to the correlation-decay time, the values of the phase variables become statistically independent.

9. Billiards

The concept of dynamical chaos was rigorously substantiated for the first time in the 1970s based on a fairly simple model of statistical physics, billiards (see Section 1). Considering billiard models traces back to studies by J Hadamard [104], who investigated motion on a twisted negative-curvature surface. Later, billiards as dynamical systems were studied by D Birkhoff [13]. A more complete analysis of the problems related to the dynamics of mass points in a bounded region was given by N S Krylov [14].

The issues that arise in analyzing billiard problems are closely related to Boltzmann's ergodic hypothesis; for this reason, billiard models still attract appreciable interest. Fairly general conditions of the onset of chaos in billiards have been obtained, and the remarkable properties of billiards have been described (see Refs [45–47, 60, 105–107] and the references therein).

In this section, we consider a simple generalization of billiard systems — billiards with oscillating boundaries. As we demonstrate below, this generalization leads to unexpected and very interesting phenomena.

Let M be the Euclidian plane. As a billiard table Q, we take a certain domain $Q \subset M$ with a piecewise-smooth boundary ∂Q. The billiard dynamical system in Q is generated by the free motion of a mass point (billiard ball) with elastic reflections from the boundary in accordance with the law 'the incidence angle is equal to the reflection angle.' If the ball reaches a corner, its further motion is not uniquely defined or is not defined at all.

As a rule, the boundary of all billiards consists of m components ∂Q_i, $i = 1, 2, \ldots, m$. If, for each component ∂Q_i, we introduce unit normals $\mathbf{n}(q)$ at each point $q \in \partial Q_i$ directed inside the domain Q, they determine the curvature $\kappa(q)$ of the curve ∂Q_i at all regular points q. The component ∂Q_i is scattering if $\kappa(q) > 0$. At $\kappa(q) = 0$ and $\kappa(q) < 0$, we respectively obtain a neutral and a focusing component of the billiard boundary. We let the union of all scattering, neutral, and focusing boundaries be respectively denoted by ∂Q^+, ∂Q^0, and ∂Q^-.

If the billiard boundary consists only of sufficiently smooth convex curves inside Q, such that $\partial Q = \partial Q^+$, then the billiard system is called a scattering (or Sinai) billiard. It is for scattering billiards that the property of mixing was proved for the first time [108]. A billiard system whose boundary consists of focusing components $\partial Q = \partial Q^-$ or neutral and focusing components $\partial Q = \partial Q^0 \bigcup \partial Q^-$ is called focusing. Under certain conditions, such a billiard can also have the property of mixing [45, 107, 109].

For arbitrary relations between the billiard-boundary parameters, the dynamics of a particle are determined by the relation between the times spent by the particle in the neighborhoods of ∂Q^+ and ∂Q^-. The exact meaning of this statement is formulated in Refs [45, 106, 109].

9.1 The Lorentz gas

A variation of the scattering billiards is a system defined in an unbounded domain D and consisting of many circular, infinitely heavy scatterers B_i with boundaries ∂Q_i, of radius R, situated at the sites of an infinite periodic triangular lattice with a step a (Fig. 17). The billiard in the domain $Q = D \setminus \bigcup_{i=1}^{r} B_i$ is called the Lorentz gas. For $\partial Q = \text{const}$, such a billiard has been studied fairly comprehensively (see, e.g., Refs [44,45, 47] and the references therein).

The properties of this model can be qualitatively different depending on the scatterer radius and the parameter a. If

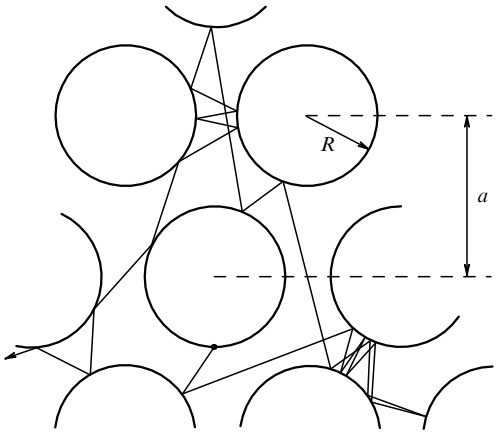

Figure 17. Two-dimensional model of the Lorentz gas. Scatterers of radius R are located at the sites of a triangular lattice with a period a.

$R \leqslant a/2$, the billiard is said to have an infinite horizon. If $R \geqslant a/\sqrt{3}$, the motion of the particle is restricted to one cell, and hence the system has a bounded horizon. If $a/2 < R < a/\sqrt{3}$, the free path of the particle is limited, but it can freely move in space. In this case, the Lorentz gas has an open horizon.

It was proved for the Lorentz gas with bounded and open horizons that the motion of the particle in a planar, two-dimensional case has the property of mixing and, moreover, reduces to Brownian motion. A positive diffusion coefficient exists in such a system [44]; it can be expressed in terms of the correlation function of the particle velocity. Thus, Brownian dynamics can follow from a purely deterministic behavior.

For an infinite horizon, the statistical properties of the Lorentz gas change because of the increasing probability of long paths. In particular, the decrease of correlations is slowed down, and the mean free path does not converge [44, 45, 110–112]. At the same time, for a Lorentz gas with a bounded or open horizon, an exponential decay of correlations is observed.

9.2 Scattering billiards with oscillating boundaries. The Fermi acceleration

The Lorentz gas is a system lying within the scope of nonequilibrium statistical mechanics. Among its variations is the model first suggested in the context of the Fermi acceleration [113, 114], in which the scatterers oscillate with a small amplitude. A generalized treatment of this problem implies a physically more natural analysis of billiards with boundaries disturbed according to a certain law. What do the oscillations of the boundaries result in? The particle in such billiards experiences both head-on and head–tail collisions with the boundary. In the first case, the reflection from the boundary occurs if the particle and the boundary move opposite to each other. In the second case, the collisions occur if the particle and the boundary move in the same direction. The head-on collisions 'heat' the particle, i.e., increase its velocity. In contrast, head–tail collisions result in 'cooling' the particle, i.e., in decreasing its velocity.

The mechanism of particle acceleration due to collisions with moving massive scatterers was first suggested by Fermi [115] to account for the origin of high-energy cosmic particles (see Ref. [116] for a review). Fermi's idea was that charged particles colliding with chaotically moving magnetic clouds in interstellar space should accelerate on average. If we regard the cloud as a massive body, we can easily understand the reasons for the acceleration. If the velocities of the clouds with which the particle collides are distributed at random, it can be said that the number of clouds moving in some direction is equal to the number of oppositely moving clouds. Therefore, the particle predominantly collides with counter-moving clouds and hence more frequently gains energy that loses it. An effective acceleration, called the Fermi acceleration, thus occurs. Later, the Fermi–Ulam and other models were developed (see Refs [54, 117–125]) to account for the origin of this phenomenon to a greater or lesser degree.

Billiards with perturbed boundaries can be considered a generalization of the Fermi–Ulam model. In particular, an investigation of the properties of elliptic and circular billiards shows [126–128] that the increase in velocity is limited in this case, as in the Fermi–Ulam model. Billiards were also considered in the domain formed by a rectangle with corners replaced with circular quadrants of radius R, with a periodically oscillating side [129]. The collision with the boundary is not absolutely elastic, the particle loses part of its velocity, and the loss is proportional to a certain constant δ ($\delta \ll 1$). This model is close to the Fermi–Ulam model but the presence of the rounded corners introduces some randomization into the particle dynamics. The relaxation of the system to an equilibrium state was studied. Similar investigations were previously carried out in Refs [130, 131] using the Fermi–Ulam model. It was shown that the quantity $\Pi(t) = E(t) - E(\infty)$, i.e., the deviation of the average energy of the system from its equilibrium value decreases exponentially in this model, $\Pi(t) \sim \exp(-t/\tau)$, which is generally natural for most physical systems. An analysis of this quantity for a billiard model [129] shows that the relaxation of the system to the equilibrium state is slower, and $\Pi(t)$ decreases in accordance with the law $\Pi(t) \sim \exp\left(-(t/\tau)^{\beta}\right)$, where $\beta < 1$ and β decreases with an increase in R.

The results in Refs [113, 114] have clarified the reasons for the decrease in the relaxation rate of the system. As the radii of the arcs are increased, the motion becomes more chaotic, which results in the acceleration of particles. Therefore, the relaxation of the system to the equilibrium, being related to the dissipation of the particle energy, is slower. Such an approach allows determining the quantity β and, accordingly, the rate of relaxation to the equilibrium for systems with predominantly chaotic dynamics.

In the general case, the situation is much more complex: the Fermi acceleration is not inherent in an arbitrary chaotic billiard with disturbed boundaries [114]. The point is that particle *deceleration* can also be observed in focusing billiards depending on the initial velocity of the particle [132]. If the initial velocity (which generally depends on the geometry of the billiard) exceeds a certain critical value, the particle is accelerated on average. On the contrary, if the initial velocity is less than the critical value, the billiard particles are decelerated. Thus, it becomes possible to separate the ensemble of particles by their velocity and decelerate particles by disturbing the boundaries. This effect of the billiard version of Maxwell's demon was first revealed and described in Refs [114, 132] based on the example of a 'stadium' billiard (i.e., a billiard whose domain Q resembles the usual stadium in shape; see Fig. 18).

The dynamics of scattering billiards with disturbed boundaries were described in detail in Refs [113, 114, 133]. For this reason, we do not consider them here, dwelling on focusing billiards. We only note that the obtained results suggest a universal presence of particle acceleration in chaotic billiards with oscillating boundaries. Moreover, based on the investigations carried out, the hypothesis that *deterministic randomness is a sufficient condition for the Fermi acceleration* was put forward [113, 114]. This assumption has been confirmed by several investigators and, in part, was recently substantiated [125, 134, 135].

9.3 Focusing billiards with oscillating boundaries. Particle deceleration

The mechanisms of the onset of chaos are different in scattering and focusing billiards [45, 47]. In the first case, a parallel beam encountering a scattering component starts diverging immediately. In the second case, after a reflection from the focusing surface, the beam converges at the focusing point. In such billiards, chaos sets in if the convergence time of the beams is shorter than the divergence time (i.e., the beam of trajectories is defocused).

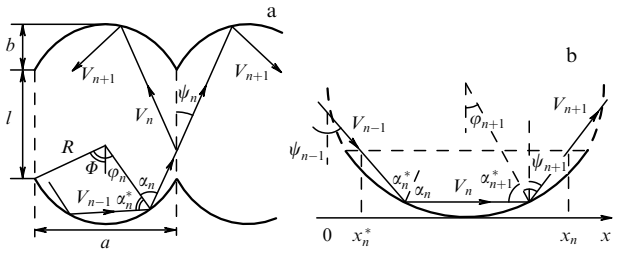

Figure 18. The construction of a map for stadium billiards.

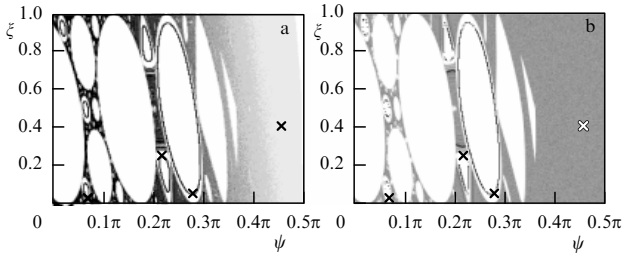

Figure 19. Phase portrait of a stadium billiard with a boundary in the form of a circular arc (a) and a parabola (b), at $a = 0.5$, $b = 0.01$, and $l = 1$ [132].

We first consider billiards with unperturbed boundaries [132]. We assume that the curvature of its focusing components is small,

$$l \gg a \gg b\,. \tag{55}$$

Let these components be formed by arcs of radius R, with the angular measure 2Φ (Fig. 18a), symmetric with respect to the vertical axis of the billiard. Geometric considerations lead to the relations $R = (a^2 + 4b^2)/8b$ and $\Phi = \arcsin(a/2R)$. In particular, the condition for chaos is satisfied for such a billiard if the arc completing the focusing component to the entire circle belongs to the domain Q (see Ref. [45]). For $b \ll a$, this yields $l/2R \approx 4bl/a^2 > 1$.

We introduce dynamic variables as shown in Fig. 18a. Let the positive direction be counterclockwise for the angles φ_n and α_n^* and clockwise for the angle α_n. If the boundary of the billiard is motionless, the incidence angle α_n^* is equal to the reflection angle α_n. Let V_n be the particle velocity and t_n the instant of the nth collision with the boundary. To construct a map describing the dynamics of the particle in such a billiard, two cases must be considered: (1) after a collision with a focusing component, the particle collides with it again (pair collisions) and (2) the next collision occurs with another focusing component.

In the case of pair collisions, a geometric analysis [132] leads to a map of the form (Fig. 18b)

$$\begin{aligned}
&\alpha_{n+1}^* = \alpha_n\,,\\
&\alpha_{n+1} = \alpha_{n+1}^*\,,\\
&\varphi_{n+1} = \varphi_n + \pi - 2\alpha_n \pmod{2\pi}\,,\\
&t_{n+1} = t_n + \frac{2R\cos\alpha_n}{V_n}\,.
\end{aligned} \tag{56}$$

If $|\varphi_{n+1}| < \Phi$, the particle continues a cascade of collisions with one component. Otherwise, the $(n+1)$th collision occurs with another focusing component. For a transition from one focusing component to another, the map can be written as

$$\begin{aligned}
&\alpha_{n+1}^* = \arcsin\left[\sin(\psi_n + \Phi) - \frac{x_{n+1}^*}{R}\cos\psi_n\right],\\
&\alpha_{n+1} = \alpha_{n+1}^*\,,\\
&\varphi_{n+1} = \psi_n - \alpha_{n+1}^*\,,\\
&t_{n+1} = t_n + \frac{R(\cos\varphi_n + \cos\varphi_{n+1} - 2\cos\Phi) + l}{V_n\cos\psi_n}\,,
\end{aligned} \tag{57}$$

where $\psi_n = \alpha_n - \varphi_n$, $x_n = (R/\cos\psi_n)\left[\sin\alpha_n + \sin(\Phi - \psi_n)\right]$, and $x_{n+1}^* = x_n + l\tan\psi_n \pmod{a}$.

Figure 19a presents a numerically obtained phase portrait of a stadium billiard specified by maps (56) and (57). As the coordinate, we chose the quantity $\xi_n = 1/2 + (R\sin\varphi_n)/a$ equal to the projection of the point of collision between the particle and the focusing component onto the $0x$ axis, normalized to the width of the billiard. The gray scale in the figure represents the number of points in the given region of phase space. To construct the graph, we used four trajectories. One of them starts in the stochastic region and includes 5×10^8 iterations. Three others start in the regular region and contain 10^7 iterations each. The initial conditions are marked with crosses in the graph.

It can be seen from Fig. 19 that stable fixed points surrounded by invariant curves exist in the system. The dynamics of particles in the neighborhood of these points are regular and are represented by motion along such invariant curves. The regions corresponding to different resonances are divided by separatrices surrounded by a stochastic layer. The width of this layer is determined by the degree of nonlinearity of the system. The particle starting its motion in such a layer visits all regions accessible to it in a chaotic manner. As the nonlinearity increases, the fixed points lose their stability; as a result, a globally stochastic domain forms, for which the entire phase space is accessible to a particle.

For an analytic treatment of map (56), (57), we make some simplifications. We approximate the focusing component of the billiard system by a segment of a parabola, $\chi(x) = 4bx(x-a)/a^2$; as the coordinates, we choose the angle ψ between the velocity vector and the vertical, $\psi \in [-\pi/2, \pi/2]$, and the projection x of the particle – boundary collision point onto the $0x$ axis, $x \in [0, a)$. With approximation (55), we then find

$$\begin{aligned}
&x_{n+1} = x_n + l\tan\psi_{n+1} \pmod{a}\,,\\
&\psi_{n+1} = \psi_n - 2\beta(x_{n+1})\,,
\end{aligned} \tag{58}$$

where $\beta(x) = \arctan\left(\chi'(x)\right)$ is the inclination angle of the tangent at the collision point. In this case, obviously, $\beta \approx 4b(2x-a)/a^2$. We now make the substitution $\xi = x/a$, $\xi \in [0, 1)$. In these variables, the map acquires the form

$$\begin{aligned}
&\xi_{n+1} = \xi_n + \frac{l}{a}\tan\psi_n \pmod{1}\,,\\
&\psi_{n+1} = \psi_n - \frac{8b}{a}(2\xi_{n+1} - 1)\,.
\end{aligned} \tag{59}$$

Clearly, one of the families of fixed points of the billiard can be written as $\{\xi = 1/2, \psi_s = \arctan(ma/l)\}$, $m = 1, 2, \ldots$. In the configuration space, these points correspond to collisions

of the particle with the center of the focusing component. We analyze the stability of such points. For this, we linearize the map using the substitution $\xi_n = \Delta\xi_n + 1/2$, $\psi_n = \Delta\psi_n + \arctan(ma/l)$. Then, an expansion into a series in $\Delta\psi$ yields

$$\Delta\xi_{n+1} = \Delta\xi_n + \frac{l}{a\cos^2\psi_s}\,\Delta\psi_n + O(\Delta\psi_n^2)\,,$$

$$\Delta\psi_{n+1} = \Delta\psi_n - \frac{16b}{a}\,\Delta\xi_{n+1}\,,$$

where $\psi_s = \arctan(ma/l)$. In this case, the transformation matrix is

$$A = \begin{pmatrix} 1 & \dfrac{l}{a\cos^2\psi_s} \\ -\dfrac{16b}{a} & 1 - \dfrac{16bl}{a^2\cos^2\psi_s} \end{pmatrix}.$$

We can easily verify that $\det A = 1$. Therefore, the obtained map preserves the measure. Hence, we can find that the condition of stability of fixed points is $\cos^2\psi_s \geqslant 4bl/a^2$, or $m^2 \leqslant l/(4b) - l^2/a^2$.

The destruction of all resonances and transition to chaos occurs if $4bl/a^2 > 1$. It can be understood without difficulty [132] that the phase-space dynamics of the billiard system in the neighborhood of a stable point are described by a twist map with the rotation number $\rho = \arccos\left[1 - 8bl/(a\cos\psi_s)^2\right]$. The time between successive collisions of the particle with the boundary is $\tau \approx l/(V\cos\psi_s)$, where V is the particle velocity. Therefore, the rotation period is

$$T_{\mathrm{rot}} = \frac{2\pi}{\rho}\,\tau = \frac{2\pi l}{\arccos\left[1 - 8bl/(a\cos\psi_s)^2\right]\cos\psi_s}\,\frac{1}{V}\,. \quad (60)$$

The phase portrait of the billiard with parabolic focusing components specified by approximate map (59) is shown in Fig. 19b. By comparing it with Fig. 19a based on the exact map (56), (57), we can see that in this case, the trajectory is uniformly distributed over the entire stochastic domain. In the first case, a higher density corresponds to the domain $\psi \geqslant 0$, while the domain $\psi \leqslant \pi/2$ is virtually empty. The difference occurs because in deriving map (59), we assumed that the depth of the focusing component b is small. As a result, if this approximation is used instead of the exact map, collisions at large ψ occur with a higher probability on the right-hand part of the arc (see Fig. 18), and hence the angle ψ decreases. Thus, the particles are pushed out to the domain of small ψ and predominantly move in the neighborhood of the regular domains.

If the boundary of the billiard is perturbed, the particle can pass from a chaotic domain to a regular domain, and vice versa. If the particle velocity is much larger than the velocity of the boundaries, it can be shown [132] that the trajectory of the representation point in the neighborhood of stable fixed points of the phase space is similar to a spiral. In this case, the rotation frequency remains the same as in the unperturbed system. In addition, it can be easily understood that a resonance between such a rotation and the perturbation of the boundaries sets in at a certain particle velocity. The presence of the resonance in the system under study leads to an unexpected result — the selection of the billiard particles according to their velocity [132]. The particles can either accelerate or decelerate depending on their initial velocity. We consider this phenomenon in more detail.

Let the focusing components of the billiard be disturbed such that the velocity of their motion is the same in magnitude at each point and directed normally to them. We also assume that the boundary oscillates according to a periodic law (Fig. 18a): $R = R_0 + r_0 f(\omega t + \eta)$, where ω is the oscillation frequency and η is the phase. Then the velocity of the boundary is $U(t) = \dot{R}$. In addition, we assume that the displacement of the boundary is small, i.e., $r_0 \ll l$. Then the map corresponding to such a billiard system is given by [114, 132]

$$\left.\begin{aligned} V_n &= \sqrt{V_{n-1}^2 + 4V_{n-1}\cos\alpha_n^* U_n + 4U_n^2}\,, \\ \alpha_n &= \arcsin\left(\frac{V_{n-1}}{V_n}\sin\alpha_n^*\right), \end{aligned}\right\} \quad (61)$$

$$\left.\begin{aligned} \alpha_{n+1}^* &= \alpha_n\,, \\ \varphi_{n+1} &= \varphi_n + \pi - 2\alpha_n \pmod{2\pi}\,, \\ t_{n+1} &= t_n + \frac{2R\cos\alpha_n}{V_n} \end{aligned}\right\} \quad \text{at} \quad |\varphi_{n+1}| \leqslant \Phi \quad (62)$$

and, if $|\varphi_n + \pi - 2\alpha_n| > \Phi$,

$$\left.\begin{aligned} \psi_n &= \alpha_n - \varphi_n\,, \\ x_n &= \frac{R}{\cos\psi_n}\left[\sin\alpha_n + \sin(\Phi - \psi_n)\right], \\ x_{n+1}^* &= x_n + l\tan\psi_n \pmod{a}\,, \\ \alpha_{n+1}^* &= \arcsin\left[\sin(\psi_n + \Phi) - \frac{x_{n+1}^*}{R}\cos\psi_n\right], \\ \varphi_{n+1} &= \psi_n - \alpha_{n+1}^*\,, \\ t_{n+1} &= t_n + \frac{R(\cos\varphi_n + \cos\varphi_{n+1} - 2\cos\Phi) + l}{V_n\cos\psi_n}\,. \end{aligned}\right\} \quad (63)$$

We note that relation (55) was not used in deriving this map. The only approximation assumed was the smallness of the displacements of the billiard boundary (i.e., we took only the leading terms into account). Expressions (62) correspond to a series of successive collisions of the particle with one focusing component and expressions (63) to a transition from one component of the boundary to another.

We analyze map (61)–(63) in two cases: for completely chaotic and nearly integrable billiards. In the first case, the billiard is a classical stadium, i.e., $\Phi = \pi/2$ (see Fig. 18), and its boundary consists of two halfcircles and two parallel straight lines tangent to them. The following parameters were chosen for this billiard: $a = 0.5$, $b = 0.25$, $l = 1$, $r_0 = 0.01$, $\omega = 1$, and $V_0 = 0.1$. The particle velocity was calculated as the average over an ensemble of 5000 trajectories with various initial conditions. The initial conditions were chosen at random at the focusing component such that the velocity vector of the particle was directed inward for the billiard table Q. As follows from a numerical analysis (curve *1* in Fig. 20a), the obtained dependence has the form $V(n) \sim \sqrt{n}$. It is interesting to note that the same dependence is also observed in the Lorentz gas [113, 114].

In the nearly integrable case, the parameter b is fairly small, the curvature of the focusing component introduces only a weak nonlinearity into the system, and relation (55) is satisfied. In such a configuration, the phase space has domains of regular and chaotic dynamics (see Fig. 19).

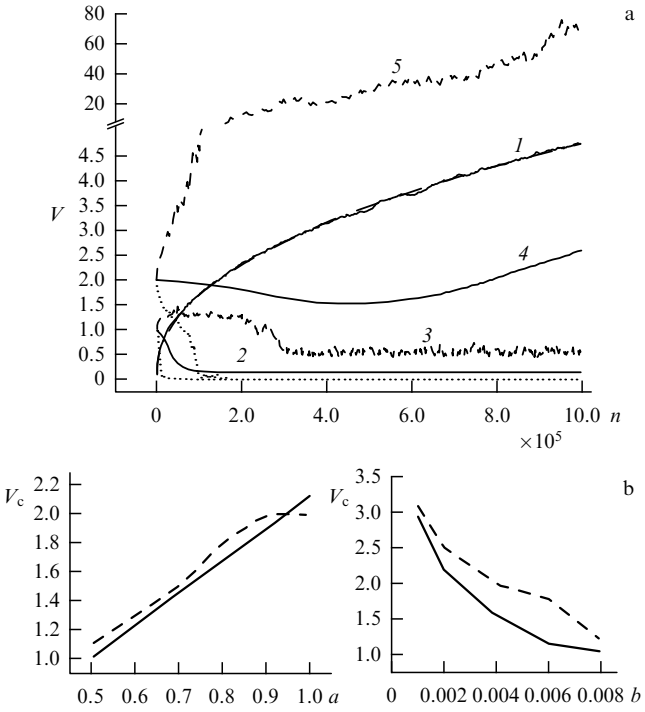

Figure 20. (a) Dependence of the average particle velocity on the number n of collisions with the boundary in a stadium billiard with oscillating walls [114, 132]. Two curves *1* correspond to a billiard with well-developed chaotic properties ($b = 0.25$). Curves *2–5* correspond to a nearly integrable billiard ($b = 0.01$), $V_0 = 1$ (curves *2* and *3*) and $V_0 = 2$ (curves *4* and *5*). (b) Dependence of the critical particle velocity V_c on the parameters a and b (see Fig. 18) in a nearly integrable stadium billiard with oscillating boundaries. Dashed curves are numerical calculations; solid curves are approximation (64) [136].

In a sufficiently small neighborhood of fixed points, the billiard system has a definite rotation period T_{rot} [see Eqn (60)]. At the same time, the oscillation period of the boundary is $T = 2\pi/\omega$. Thus, at a certain velocity V_r, a resonance is observed in the system. As follows from the analysis in Ref. [132], the dependence of the particle velocity on the number of collisions is different for the two sides of the resonance. For the initial velocity $V_0 < V_r$, the particle velocity decreases to the final value $V_{\text{fin}} < V_r$ and the velocity distribution of particles in the interval $(0, V_{\text{fin}})$ approaches a stationary distribution. But if $V_0 > V_r$, the particle velocity can reach high values. In this case, the distribution of particles is time-dependent and increases indefinitely.

Figure 20a (curves *2–5*) shows the dependence of the particle velocity on the number of collisions. Based on 5000 realizations for each initial velocity, three curves were plotted for the velocities (mean, minimum, and maximum) achieved by the nth collision with the boundary. The range of velocity variation can be determined thus. As follows from the figure, at $V_0 < V_r$, the average particle velocity (curve *2*) gradually decreases and approaches a constant. The maximum particle velocity (curve *3*) decreases to V_{fin} and fluctuates around this value. In the case where $V > V_r$, the minimum particle velocity also decreases. This means that particles that have found their way to the small-velocity domain are also present in the ensemble. According to the results of numerical analysis, the fraction of such particles is about 75%. At the same time, particles with fairly high velocities (curve *5*) corresponding to the maximum velocity in the ensemble are present. As a result, the average velocity (curve *4*) increases.

Therefore, a certain critical velocity exists in this billiard system; particles with velocities below the critical value decelerate and those with velocities above the critical value accelerate. The critical velocity can be expressed as

$$V_c = \frac{\omega l}{\cos\psi_s \arccos\left[1 - 8bl/(a\cos\psi_s)^2\right]} . \tag{64}$$

Figure 20b shows the dependence of critical velocity (64) on the characteristic parameters of the billiard, the width a and the depth b of the focusing component (Fig. 18a). The dashed curves represent the dependences found numerically and the solid curves are the approximation based on Eqn (64). It can be seen that the critical velocity approaches zero with an increase in b, i.e., particle deceleration is not observed in the corresponding billiards. The same is the case if the parameter a decreases. This is because such variations in the parameter increase the nonlinearity in the system and, accordingly, the degree of chaotization in the billiard dynamics.

Thus, in the billiard system considered, the particles in the ensemble are separated in accordance with their velocity: some of them are accelerated and others are decelerated.

10. Conclusion

Theoretical analysis of nonlinear systems with complex, particularly chaotic, behavior is among the most important areas of modern physics. This is because chaotic processes are inherent in a very wide class of phenomena. In addition, nonlinear dynamics, in the framework of which the concept of deterministic chaos is developed, combines extremely diverse applications suggested by the natural sciences, from the theory of excitable media to the problems of information processing and storage [137]. The nature and consequences of chaos remain among the most challenging issues in these fields.

Substantial progress has been achieved by now in the investigation of classical mechanics systems. In particular, a possibility for renormalization was found for the resonances of perturbed motion near the separatrix [138, 139]. Notable advances were achieved in the KAM theory [53, 70]; its generalizations to infinite-dimensional systems were found [140, 141]; results concerning the presence of diffusion in systems described by partial differential equations were obtained [142, 143]; consequences of the KAM theory for reversible systems were extracted [144]; and quantum versions of the KAM theory were developed. For diffusion trajectories, a structure was discovered that came to be known as the 'stochastic web' [60, 62, 145].

Among the areas in which the theory presented here finds direct application, The author is most familiar with the N-body problem, celestial mechanics, and systems of classical mechanics; we therefore outline the range of issues that belong precisely to these areas. Applications to other fields are described in some publications in the reference list.

In his time, Poincaré called the N-body problem a fundamental unresolved problem of classical mechanics. V M Alekseev substantially contributed to the development and investigation of this problem [146]. The KAM theory cannot be directly used to analyze planetary problems,

because proper degeneration is present there. But generalizations are known [23, 147] that allow applying the results of the KAM theory to certain systems from a relatively narrow class.

In the past 25 years, some extremely important results were obtained in celestial mechanics: the Painlevé hypothesis on collisionless singularities was proved [148 – 150], new remarkable solutions (choreographies) of the N-body problem were found (see Ref. [151]), and resonances were proved to play a key role in the dynamics of celestial bodies [152]. In addition, large-scale chaos was revealed in the Solar system, which seems to be among the most important factors responsible for the formation of the asteroid belt and for the diffusion of comets from the outer regions of the Solar system [153 – 155] (astronomical implications of this phenomenon are described in Refs [156, 157]). It was also established that chaos is present in the system of Saturn's satellites [158]. We mention the notable result that the Arnol'd diffusion is present in the three-body problem [159, 160] (see Ref. [161] for a discussion of the diffusion in the N-body problem). Reference [162] presents a proof of the Arnol'd diffusion in the D'Alembert's problem of the rotation of an oblate planet around a motionless star. A recently published collection [163] presents some new results that refer to the development of chaos in celestial mechanics and the N-body problem.

The problem of migration of minor bodies in the Solar system is among the incompletely resolved problems of celestial mechanics closely related to instability and chaos (see Refs [54, 154, 164, 165] and the references therein); in particular, these are the formation of Kirkwood gaps [166] (i.e., gaps in the distribution of asteroids near resonances with Jupiter's mean motion) [167 – 169], of the Kuiper (or Edgeworth – Kuiper) belt [170, 171], and of the Oort cometary cloud [172]. Closely related to these issues is the asteroidal-hazard problem (highly popular in recent years), i.e., the evaluation of the probability of migration of minor bodies into the terrestrial-planet zone [164, 173, 174]. We also note theoretical issues of the finiteness of the number of relative equilibria in the classical N-body problem and of collisional trajectories.

Among the very interesting phenomena that call for explanation and are related to the theory of chaotic dynamical systems, we note certain complex configurations of the major-planet rings [164, 175]. In recent years, the interest in this problem was revived by the *Cassini* mission.

The stability of the Solar system remains an unresolved fundamental problem of celestial mechanics, which has commanded investigators' attention for a long time [164, 176, 177]. Recent studies have shown that although the Solar system is not stable, it can nonetheless be considered stable over time intervals shorter than its lifetime (about five billion years). Within several billion years, the orbital diffusion of terrestrial planets can result in the recession of Mercury or its collision with Venus (see Ref. [178]). At the same time, the motion of major planets is to be fairly regular. We note, however, that many other effects unaccounted for in the Newton equations (frictional interaction with interstellar dust, relativistic corrections, etc.) must manifest themselves on such a time scale. Therefore, the question of the stability of the Solar system in the formulation in question is only indirectly related to the actual planetary motion.

Billiard-type systems with perturbed boundaries described in Section 9 belong to a relatively fresh area of mathematical physics, which opens wider horizons in investigating certain fundamental problems of classical statistical mechanics (see Ref. [137]). In particular, it presents a new view of the problem of cosmic-particle acceleration to high energies, the Fermi acceleration. Ever since the classic study by Fermi [115], this problem has attracted the attention of researchers in various branches of physics — optics [179, 180], plasma physics [181, 182], astrophysics [183 – 185], etc. Similar ideas were recently invoked to account for some new experimental results obtained in atomic physics [186].

Specially configured billiards are also used to study the phenomenon of Fermi acceleration [113, 114, 134]. Such systems have the distinctive feature that they could be used, after a suitable modification, in experimental investigations [187 – 189].

The authors of Refs [113, 114], based on their results, put forward the hypothesis that if a billiard with a fixed boundary is chaotic, switching on a disturbance of its boundary leads to the Fermi acceleration. This hypothesis was tested and partially confirmed (see Refs [125, 134, 135]).

In focusing nearly integrable billiards for which the curvature of the boundary components introduces only a weak nonlinearity into the system, periodic disturbances of the boundaries lead to an interesting new phenomenon, particle separation by velocity. Depending on the initial velocity, particles in the billiard system either accelerate or decelerate. In the case of acceleration, the particle distribution is time-dependent and increases unlimitedly. For decelerating particles, an equilibrium velocity distribution is established. This property of a billiard system can be regarded as a billiard version of Maxwell's demon: the periodic perturbations of the boundaries result in velocity losses by slow ('cold') particles, to a certain nonzero level, and the acceleration of fast ('hot') particles. The physical mechanism decelerating the particles is not yet completely understood [136].

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