Stabilized Chaos in the Sitnikov Problem

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1 Formulation of the Problem

The Sitnikov problem consists of two equal masses M (called primaries) moving in circular or elliptic orbits about their common center of mass and a third, test mass μ moving along the straight line passing through the center of mass normal to the orbital plane of the primaries. This problem has attracted the attention of many other authors (see for instance [1–9]). The equation of motion can be written in scaled coordinates and time as

$$\ddot{z} + \frac{z}{[\rho(t)^2 + z^2]^{3/2}} = 0, \tag{1}$$

where z denotes the position of the particle μ along the z-axis and $\rho(t) = 1 + e\cos(t) + O(e^2)$ is the distance of one primary body from the center of mass. Here we see that the system (1) depends only on the eccentricity, e, which we shall assume to be small. The linear approach to this system with assumption that (1) possesses moderate eccentricity and small amplitudes was carried out in [7]. We first consider the circular Sitnikov problem i.e. when $e = 0$, for which $H = \dfrac{1}{2}p^2 - \dfrac{1}{\sqrt{1+z^2}}, p = \dot{z}$. The level curves $H = h$, where $h \in [-2, +\infty)$, partition the phase space (p, z) into qualitatively different types of orbits. We are interested in solutions that correspond to the level curves $H = 0$, namely two parabolic orbits that separate elliptic and hyperbolic orbits and can be considered as a separatrix between these two classes of behavior. To make clear how this problem is related to homoclinic orbits, let us employ the non-canonical transformation [9] $z = \tan u, p = \dot{z}, u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], v \in R$. Then the Hamiltonian for (1) in the new variables (u,p) has the form $H(u,p) = \dfrac{1}{2}p^2 - \dfrac{1}{\rho(t)^2 + \tan^2 u} = H_0(u,p) + eH_1(u,p,t,e)$, where $H_0(u,p) = \dfrac{1}{2}p^2 - \cos u$. One can see that when $e = 0$ the form of the Hamiltonian that obtained after

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non-canonical transformation exhibits the pendulum character of motion. In this work we consider only small values of e. Hence due to the KAM-theorem, since our system has 3/2 degrees of freedom the invariant tori bound the phase space and chaotic motion is finite and takes place in a small vicinity of a separatrix layer. Our analysis is directed to the stabilization of this chaotic behavior in the elliptic Sitnikov problem.

2 Stabilization of Chaotic Behavior in the Extended Sitnikov Problem

The idea that chaos may be suppressed goes back to the publications. [10–14] We consider the problem of stabilization of chaotic behavior in systems with separatrix contours that can be described by (2.1) $\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon \mathbf{f}_1(\mathbf{x}, t)$, where $\mathbf{f}_0(\mathbf{x}) = (f_{01}(\mathbf{x}), f_{02}(\mathbf{x}))$, $\mathbf{f}_1(\mathbf{x}, t) = (f_{11}(\mathbf{x}, t), f_{21}(\mathbf{x}, t))$. For this equation the Melnikov distance, which "measures" (in the first order of ε) the distance between stable and unstable manifolds (Fig. 1), $D(t_0)$ is given by $D(t_0) = -\int\limits_{-\infty}^{\infty} \mathbf{f}_0 \wedge \mathbf{f}_1 dt \equiv I[g(t_0)]$. We assume that $D(t_0)$ changes its sign. To suppress chaos we should get *a function of stabilization* $\mathbf{f}^*(\omega, t)$ that leads to a situation when separatrices are not intersected: (2.2) $\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon [\mathbf{f}_1(\mathbf{x}, t) + \mathbf{f}^*(\omega, t)]$, where $\mathbf{f}^*(\omega, t) = (f_1^*(\omega, t), f_2^*(\omega, t))$. Suppose $D(t_0) \in [s_1, s_2]$ and $s_1 < 0 < s_2$. After the stabilizing perturbation $\mathbf{f}^*(\omega, t)$ is applied we have two cases: $D^*(t_0) > s_2$ or $D^*(t_0) < s_1$, where $D^*(t_0)$ – Melnikov distance for system (2.2). We consider the first case (analysis for the second one is similar). Then $I[g(t_0)] + I[g^*(\omega, t_0)] > s_2$, where $I[g^*(\omega, t_0)] = -\int\limits_{-\infty}^{+\infty} \mathbf{f}_0 \wedge \mathbf{f}^* dt$. This expression is true for all left hand

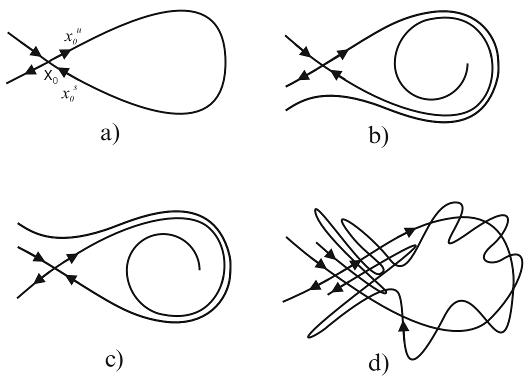

Fig. 1. Poincaré section $t = \text{const} \pmod T$ of the system (2.1) for $\varepsilon = 0$ (**a**) and $\varepsilon \neq 0$ (**b**–**d**). Only in case of (**d**) we have homoclinical chaos

side values of inequality that is greater than s_2. It is derived that $I[g(t_0)] + I[g^*(\omega, t_0)] = s_2 + \chi = const$, where $\chi, s_2 \in \mathbb{R}^+$. Therefore $I[g^*(\omega, t_0)] = \text{const} - I[g(t_0)]$. On the other hand, $I[g^*(\omega, t_0))] = -\int_{-\infty}^{\infty} \mathbf{f}_0 \wedge \mathbf{f}^* dt$. We choose $\mathbf{f}^*(\omega, t)$ from the class of functions that are absolutely integrable on an infinite interval such that they can be represented in Fourier integral form. Then $\mathbf{f}^*(\omega, t) = \text{Re}\{\hat{A}(t)e^{-i\omega t}\}$. Here we suppose that $\hat{A}(t) = (A(t), A(t))$ i.e. the regularizing perturbations applied to both components of (2.2) are identical. Therefore $-\int_{-\infty}^{\infty} \mathbf{f}_0 \wedge \left\{\hat{A}(t)e^{-i\omega t}\right\} dt = \text{const} - I[g(t_0)]$. The inverse Fourier transform yields: $\mathbf{f}_0 \wedge \hat{A}(t) = \int_{-\infty}^{\infty} (I[g(t_0)] - \text{const})\, e^{i\omega t} d\omega$. Hence,

$A(t) = \dfrac{1}{f_{01}(x) - f_{02}(x)} \int_{-\infty}^{\infty} (I[g(t_0)] - \text{const})\, e^{i\omega t} d\omega$. Here $A(t)$ can be interpreted as the amplitude of the "stabilizing" perturbation. Thus, for system (2.1) the external stabilizing perturbation has the form:

$f^*(\omega, t) = \text{Re}\left[\dfrac{e^{-i\omega t}}{f_{01}(x) - f_{02}(x)} \int_{-\infty}^{\infty} (I[g(t_0)] - \text{const})\, e^{i\omega t} d\omega\right]$. In conservative case const=0. From the physical point of view the dynamics of the chaotic system are stabilized by a series of "kicks". The orbit that was chaotic and became regular under influence of the external perturbation we call the *stabilized orbit.*

Let us now consider two bodies of mass m that are placed in the vicinity of the stable triangular Lagrange points of the Sitnikov problem (as shown in Fig. 2). Here we treat only the hierarchical case : $\mu \ll m \ll M$. In the new

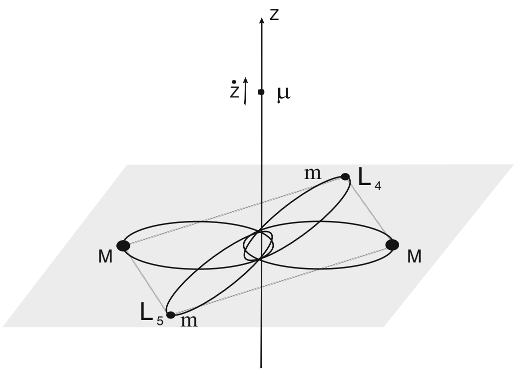

Fig. 2. Geometry of the Extended Sitnikov problem

configuration we can achieve the situation when the influence of bodies placed close to the triangular Lagrange points to the particle μ can be presented as a series of periodic impulses. Taking into account the new configuration (Fig. 2) we may say that there is a connection between the extended elliptical Sitnikov problem and the motion of the chaotic nonlinear pendulum with an external impulse-like perturbation. The Hamiltonian of such system changes to

$$H(u,p) = H_0(u,p) + e\left[H_1(u,p,t,e) + \sum_{n=-\infty}^{+\infty} \delta(t - n\tau)\right], \tag{2}$$

where τ is the duration of the impulsive forces that the particle μ experiences from bodies in the vicinity of L_4 and L_5. Now taking into account the result from the first part of this section we conclude that the forces which the particle experiences from bodies in the neighborhood of L_4 and L_5 act on the chaotic behavior of μ as an external stabilizing perturbation and the system (2) represents the system with stabilized chaotic behavior that corresponds to the stabilized orbits in the extended Sitnikov problem. The extension of the analysis carried out above to the corrections of higher order in ε of the (1) and numerical verification of the obtained results could be found in [16].

In summary, on the basis of the elliptic Sitnikov problem we constructed a configuration of five bodies which we called the extended Sitnikov problem and analytically showed that in this configuration along with chaotic and regular orbits a new type of orbit (stabilized) could be realized. We thank Carles Simó and David Farrelly for valuable discussions. A. Dzhanoev acknowledges that this work is supported by the Spanish Ministry of Education and Science under the project number SB2005-0049. Financial support from project number FIS2006-08525 (MEC-Spain) is also acknowledged. This work was supported in part by the Cassini project.

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