Stabilization of Chaotic Behavior in the Restricted Three-Body Problem

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Abstract. A new type of orbit in the restricted three-body problem is constructed. It is analytically shown that along with the well known chaotic and regular orbits in the three-body problem there also exists a qualitatively different type of orbit which we call "stabilized." The stabilized orbits are a result of additional orbiting bodies that are placed in the triangular Lagrange points. The results are well confirmed by numerical orbit calculations.

Keywords: Restricted three-body problem, Sitnikov problem, chaos, separatrix splitting, stabilized orbits.
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INTRODUCTION

More than a century ago Henri Poincaré suggested that highly complex behavior could occur in the three-body problem. Later, for the restricted three-body problem it was analytically verified that the complex behavior is due to existence of transverse heteroclinic points. A well-known example of the chaoticity of the restricted three-body problem is the Sitnikov problem.

The Sitnikov problem consists of two equal masses M (called primaries) moving in circular or elliptic orbits about their common center of mass and a third, test mass μ moving along the straight line passing through the center of mass normal to the orbital plane of the primaries

The circular problem was considered first by McMillan in 1913 [1],who found the exact solution of the equations of motion for vanishing eccentricity and showed that it can be expressed in terms of elliptic integrals. Detailed discussion of this case can be found in Stumpff [2]. This problem became more important when Sitnikov [3] in 1960 investigated the elliptic case ($e > 0$) and proved the possibility of the existence of oscillatory motions which were earlier predicted by Chazy in 1922-32. Alekseev [4] in 1968-69 proved that in the Sitnikov problem all possible combinations of final motions in the sense of Chazy are realized. Later in 1973 an alternative proof of Alexeev's results was carried out by Moser [5]. Since then the Sitnikov problem has attracted the attention of many other authors. Interesting qualitative work was done by Llibre and Simó [6] in 1980 and later by C.Marchal [7] in 1990. Hagel [8] derived an approximate solution of the equation of motion for particle μ in action-angle variables. A numerical study of the great variety of possible structures in phase space for the Sitnikov problem has been done by Dvorak [9]. Using Melnikov's method Dankowicz and Holmes [10] were able to show the existence of transverse homoclinic orbits. They proved that for all but a finite

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number of values of the eccentricity e the system is non-integrable, i. e., chaotic.

The main objective of this paper is to study through analytical and numerical methods the problem of stabilization of chaotic behavior in this special restricted three-body problem.

FORMULATION OF THE PROBLEM

The equation of motion can be written, in scaled coordinates and time as

$$\ddot{z}+\frac{z}{[\rho(t)^2+z^2]^{3/2}}=0, \tag{1}$$

where z denotes the position of the particle μ along the z-axis and $\rho(t)=1+e\cos(t)+O(e^2)$ is the distance of one primary body from the center of mass. Here we see that the system (1) depends only on the eccentricity, e, which we shall assume to be small.

We first consider the circular Sitnikov problem, i.e., when $e=0$, for which

$$H=\frac{1}{2}v^2-\frac{1}{\sqrt{1+z^2}} \qquad v=\dot{z}. \tag{2}$$

The level curves $H=h$, where $h\in[-2,+\infty)$, partition the phase space (v,z) into qualitatively different types of orbits as shown in Figure 1. We are interested in solutions

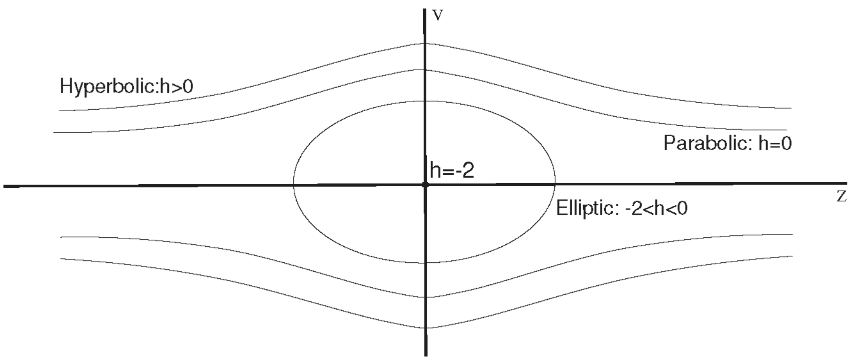

FIGURE 1. Phase plane for the unperturbed Sitnikov problem

that correspond to the level curves $H=0$, namely two parabolic orbits that separate elliptic and hyperbolic orbits and can be considered as a separatrix between these two classes of behavior. Further from equation (1) it is easy to show that there are just two fixed points: (0,0) – the center at the origin and $(\pm\infty,0)$. Making use of the McGehee transformation [10] the fixed points at $(\pm\infty,0)$ correspond to hyperbolic saddle points. Then taking into account that parabolic orbits act as connections or heteroclinic orbits between these two fixed points one may conclude that the stable and unstable manifolds of saddles correspond to the parabolic orbits.

To make clear how this problem is related to heteroclinic orbits, let us employ the non-canonical transformation [10]:

$$z = \tan u, v = \dot{z}, u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], v \in R. \tag{3}$$

Then the Hamiltonian for the equation (1) in the new variables (u,v) has the form:

$$H(u,v) = \frac{1}{2}v^2 - \frac{1}{\rho(t)^2 + \tan^2 u} = H_0(u,v) + eH_1(u,v,t,e), \tag{4}$$

where $H_0(u,v) = \frac{1}{2}v^2 - \cos u$. One can see that when $e = 0$ our system reduces to a nonlinear pendulum.

Based on this connection between the dynamics of the nonlinear pendulum and the Sitnikov problem one can show that if $e \in (0,1)$ then for all but a possibly finite number of values of e in any bounded region, the system (1) is chaotic [10]. In this work we consider only small values of e. Hence due to the KAM-theorem, since our system has 3/2 degree of freedom the invariant tori bound the phase space and chaotic motion is finite and takes place in a small vicinity of a separatrix layer.

As mentioned in the introduction, our analysis is directed to the stabilization of this chaotic behavior in the elliptic Sitnikov problem. In general, this problem is related to the stabilization and control of unstable and chaotic behavior of dynamical systems by external forces. A comprehensive study of chaotic systems with external controls was done in [16]. In the next section we will give a brief review of these results.

STABILIZATION OF CHAOTIC BEHAVIOR

Most nonlinear dynamical systems possess chaotic behavior for a certain choice of parameters. Since there are situations for which this behavior might be undesirable, different methods have been developed in the past years to suppress or control chaos.

The idea that chaos may be suppressed goes back to the publications [11, 12] where it has been proposed to perturb periodically the system parameters. Later this idea has been analytically verified [13]. The method of controlling chaos has been introduced in the paper [14] (the history of this question see in review [15]).

The main idea of this section is to show how to find a general perturbation which applied to the dynamical system in such a way that the dynamical system finally shows no chaotic behavior. In other words, given a certain dynamical system for which chaos exists for a given choice of parameters, the challenge is to find an appropriate perturbation, that we call the “function of stabilization”, which would make the dynamical system non-chaotic. We consider an application of the Melnikov method to the analysis of the chaos suppression phenomenon in systems with separatrices. Such an approach allows us to find an analytical expression of the perturbations for which the Melnikov distance $D(t_0)$ does not change sign (see also [16]), suppressing the chaotic behavior and thus stabilizing the orbits of the system.

We analyze the problem of stabilization of chaotic behavior in systems with separatrices that can be described by the equation

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon \mathbf{f}_1(\mathbf{x},t), \tag{5}$$

where $\mathbf{f}_0(\mathbf{x}) = (f_{01}(\mathbf{x}), f_{02}(\mathbf{x}))$, $\mathbf{f}_1(\mathbf{x},t) = (f_{11}(\mathbf{x},t), f_{21}(\mathbf{x},t))$. For this equation the Melnikov distance $D(t_0)$ is given by $D(t_0) = -\int_{-\infty}^{\infty} \mathbf{f}_0 \wedge \mathbf{f}_1 dt \equiv I[g(t_0)]$. Let us assume that $D(t_0)$ changes sign. To suppress chaos we should get *a function of stabilization* $\mathbf{f}^*(\omega,t)$ that leads us to a situation where the separatrices do not intersect:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon \left[\mathbf{f}_1(\mathbf{x},t) + \mathbf{f}^*(\omega,t)\right], \tag{6}$$

where $\mathbf{f}^*(\omega,t) = (f_1^*(\omega,t), f_2^*(\omega,t))$.

Omitting technical details we write here the external stabilizing perturbation for the system (5) [16]:

$$f^*(\omega,t) = \mathrm{Re}\left[\frac{e^{-i\omega t}}{f_{01}(x) - f_{02}(x)} \int_{-\infty}^{\infty} \left(I\left[g\left(t_0\right)\right] - \mathrm{const}\right) e^{i\omega t} d\omega\right]. \tag{7}$$

In [16] a function of stabilization was obtained for a case that is useful for many physical applications, i.e., when the dynamics of the systems that admit an additive shift from the critical value of the Melnikov function $D(t_0)$ are regularized by the perturbation:

$$f^*(\omega,t) = -\frac{4\pi a \delta(t)}{f_{01}(x) - f_{02}(x)} \cos(\omega t), \tag{8}$$

where $\delta(t)$ is a Dirac delta–function.

From the physical point of view the dynamics of the chaotic system are stabilized by a series of "kicks".

EXTENDED SITNIKOV PROBLEM

In the vicinity of the orbiting primaries there exist five equilibrium points lying in the $z = 0$ plane. The points L_1, L_2, L_3 are unstable and collinear with the primaries, while each of L_4 and L_5 forms an equilateral triangle with the primaries and are stable, depending on the mass of the primaries.

Let us now consider two bodies of mass m that are placed in the stable triangular Lagrange points of the Sitnikov problem (see Figure 2). Here we treat only the case when the masses m satisfy the condition: $M \gg m \gg \mu$. In the new configuration that constiutes the extended Sitnikov problem a particle of mass μ experiences forces from the primaries and masses m in L_4 and L_5. These forces are perpendicular to the primaries plane and therefore the particle motion remains on the z axis. Since the bodies of mass m orbit around their common center of mass, their distance ρ' alternates between ρ'_{min}

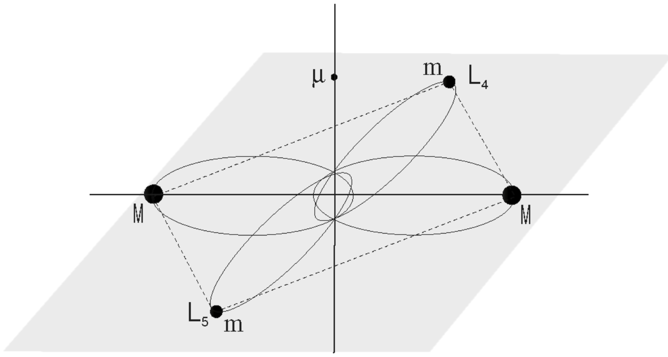

FIGURE 2. Stabilization of chaotic behavior of particle μ

(periastron) and ρ'_{max} (apastron), consequently the forces between these bodies and the particle μ increase in a close encounter to the barycenter and vanish as the bodies move away from z axis. So we can achieve the situation when the influence of bodies that are placed in triangular Lagrange points to the particle μ can be presented as a series of periodic impulses.

Recall that at the end of Section 2 we showed how the elliptical Sitnikov problem deals with heteroclinic orbits that lead to non-integrability due to the existence of transverse heteroclinic points. It was done through the connection between this problem and the pendular character of motion described by (4). Therefore taking into account the new configuration of the restricted three-body problem (Figure 2) we may say that there is a connection between the extended elliptical Sitnikov problem and the motion of the chaotic nonlinear pendulum with an external impulse-like perturbation. The Hamiltonian of such system changes to

$$H(u,v) = H_0(u,v) + e\left[H_1(u,v,t,e) + \sum_n \delta(t - n\tau)\right], \tag{9}$$

where τ is the duration of the impulsive forces that the particle μ experiences from bodies at L_4 and L_5.

Now taking into account the result (8) of Section 3 we conclude that the forces which the particle experiences from bodies in L_4 and L_5 act on the chaotic behavior of μ as an external stabilizing perturbation and the system (9) represents the system with stabilized chaotic behavior that corresponds to the stabilized orbits in the extended Sitnikov problem.

NUMERICAL RESULTS

As mentioned before, in the circular Sitnikov problem when $e = 0$ the phase space is partitioned into invariant curves corresponding to different energies. For $e > 0$ this structure is broken. This is apparent for eccentricity $e = 0.07$ in Figure 3: just a few

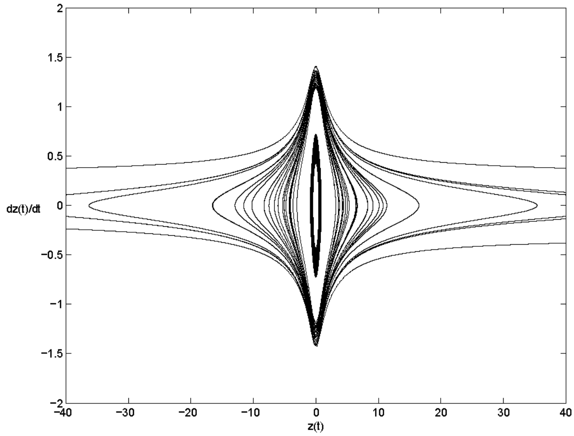

FIGURE 3. Phase portrait of the particle motion in Sitnikov problem for $e = 0.07, M = 0.5$

invariant curves survive. In this figure we also can see hyperbolic and parabolic orbits that corresponds to energy $h \geq 0$. These orbits escape to infinity with positive or zero speed respectively. Now, if we consider the extended Sitnikov problem then one can see (Figure 4) that all orbits are in a bounded region and there are no escape orbits. We have carried out computer simulations with the following mass values: $M = 0.5$, $m = 0.17$.

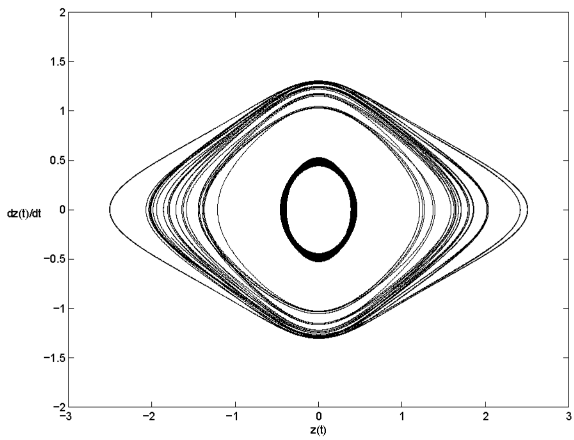

FIGURE 4. Phase portrait of the stabilized particle orbits in the extended Sitnikov problem for $e = 0.07, M = 0.5, m = 0.17$

So one may infer that stabilized orbits of the pendulum system with Hamiltonian (9) correspond to the stabilized orbits in the extended Sitnikov problem, thus confirming the conclusion of the previous section.

CONCLUSIONS

We have performed a study of the existence of a qualitatively different from known before type of orbits in the restricted three-body problem: the stabilized orbits. On the basis of the elliptic Sitnikov problem we constructed a configuration of five bodies which we called the extended Sitnikov problem and showed that in this configuration along with chaotic and regular orbits the stabilized orbits could be realized.

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