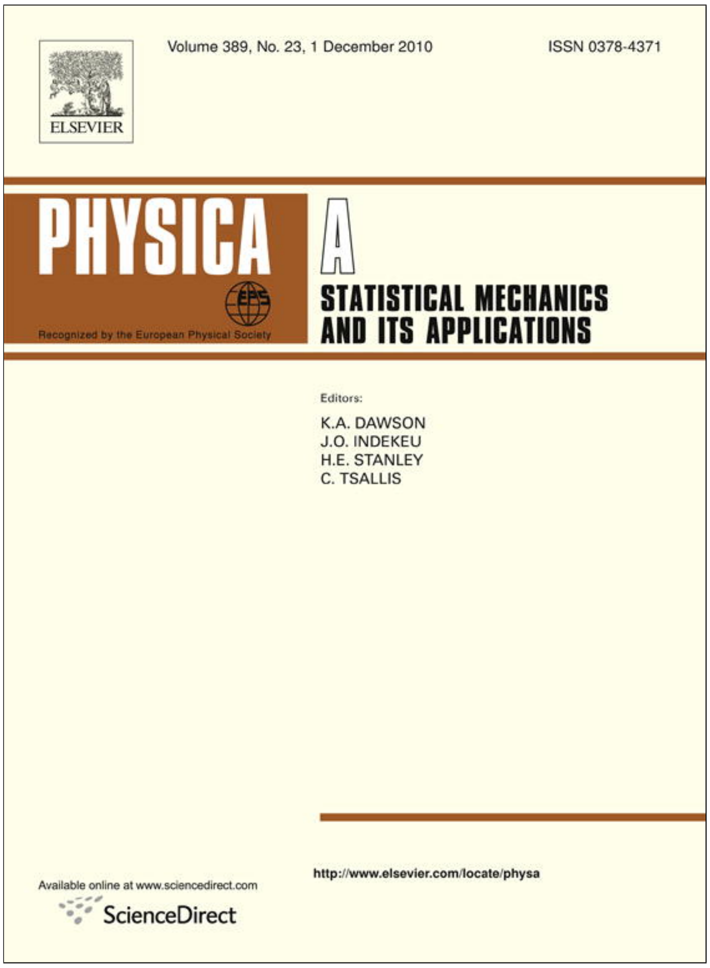
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Separation of particles in time-dependent focusing billiards

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A B S T R A C T

We consider a family of stadium-like billiards with time-dependent boundaries. Two different cases of time dependence are studied: (i) the fixed boundary approximation and (ii) the exact model which takes into account the motion of the boundary. It is shown that when the billiards possess strong chaotic properties, the sequence of their boundary perturbations is the Fermi acceleration phenomenon which is three times larger than in the case of the fixed boundary approximation. However, weak mixing in such billiards leads to particle separation. Depending on the initial velocity three different things occur: (i) the particle ensemble may accelerate; (ii) the average velocity may stay constant or (iii) it may even decrease.

1. Introduction

Mathematical planar billiards symbolize a mass point moving freely in a table with elastic reflections from the boundary. The table is a bounded region of the plane, and its shape can be arbitrary. The billiard particle moves along straight lines with a constant velocity and reflects from the boundary according to the law: "the angle of reflection is equal to the angle of incidence". The particle dynamics of such billiard systems is completely determined by the boundary property.

The contemporary studies of billiards were initiated by Sinai in 1962 in connection with the foundation of the Boltzmann ergodic hypothesis [1]. Later in his remarkable paper [2] the class of dispersing billiards (now also called as the Sinai billiards) has been introduced. Such billiards are ergodic with mixing, and they are K-systems. In the following years the other famous chaotic billiards have been constructed. In particular, Bunimovich [3] described an example of a billiard with neutral and focusing components—the renowned Bunimovich stadium.

A remarkable result was proven in Ref. [4] that typical trajectories in the periodic Lorentz gas (which is in fact a billiard) converge, after appropriate rescaling of time and space, to the motion of Brownian particles. This implies that the billiard particles may have a deterministic diffusive behavior.

A natural generalization of the static billiards is to consider billiards when their boundaries are perturbed with time (see Ref. [5] and the references cited therein). Really, the Lorentz gas corresponds to electron flow through an array of heavy scatterers (ions) embedded at the sites of an infinite lattice. Therefore, in general, all scatterers should sway randomly near their equilibrium state. Moreover, in many physical branches time-dependent billiard models are used to describe some important phenomena (see, e.g., Ref. [6] and the references cited therein).

When the billiard boundary is time dependent, the velocity of the particles may change at each collision. Depending on whether the billiard boundary is approaching or receding, the particle gains or loses energy, respectively. Thus, an urgent question for these systems is: on average, will particles accelerate or moderate? The phenomenon of the unbounded energy gain of a flow of particles under the action of time-dependent external forces is known as Fermi acceleration. It was first described by Fermi [7] to explain the origin of high-energy cosmic rays. He assumed that charged particles, which collide

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with chaotically moving magnetic clouds in interstellar space, should accelerate on average. If we regard the cloud as a massive body, one can easily understand the reason of acceleration. If the velocity directions of the clouds are distributed at random, one may say that the number of clouds moving in some direction is equal to the number of oppositely moving clouds. Therefore, the particle predominantly collides with counter-moving clouds and hence more frequently gains energy than loses it. Thus, the average velocity grows, leading to Fermi acceleration.

A number of models which were appealed to describe this phenomenon in detail has been proposed (see Refs. [8–13] and the references therein). These models explain the origin of Fermi acceleration to a greater or lesser degree. So, for the simplest 1D configuration when the particle moves freely between oscillating and fixed walls (Fermi–Ulam model) it was shown that if the oscillation phase at the moment of collision is a random value, the particle can acquire an infinitely high velocity. In the case of smooth time dependence of the wall motion, the invariant spanning curves prevent the growth of the particle velocity that forbids Fermi acceleration [10,12].

In chaotic billiards, even if the boundary velocity is a smooth function of time, the incidence angle of a particle to the boundary can be treated as a random quantity. Therefore, the normal velocity component at the collision point is assumed to be a stochastic value. On the basis of this observation the authors of paper [14] advanced a conjecture that Fermi acceleration in time-dependent billiards will be observed if the corresponding static (unperturbed) billiards possess chaotic dynamics.

As was found by dynamical approach, this conjecture is valid for the Lorentz gas [15], for Bunimovich' stadium [16], annular billiards [17,18] and for a family of oval billiards [19]. Lately, using the theory of dynamical systems Fermi acceleration in non-autonomous billiard-like systems has been investigated [20,21]. Recently, in spite of the widespread assumption that driven elliptical billiard cannot exhibit Fermi acceleration, this phenomenon has been detected [22]. In such billiards the destruction of adiabatic invariants near separatrices takes place, such that if initial conditions are chosen in this region, the particle acceleration will be observed. However, the acceleration value turns out to be much less than that for non-integrable billiards. Finally, applying thermodynamic methods it was shown [23] that Fermi acceleration is inherent in the Lorentz gas of a quite general configuration.

In general, however, the situation is much complex: not for all initial particle velocities in chaotic billiards Fermi acceleration is observed. Moreover, under certain conditions focusing billiards may even moderate particles [16]. In this case particles in the ensemble are separated in accordance with their initial velocities: some of them are accelerated and others are decelerated. This property of a billiard system can be treated as a billiard version of Maxwell's demon [24]: slow ("cold") particles lose their velocity up to a certain nonzero level, whereas the velocity of fast ("hot") particles grows.

This unexpected effect has been obtained in a fixed boundary approximation (FBA) [16,24] which neglects boundary displacement. However, as was shown in papers [25,26], this approach leads to underestimation of the energy gain. Therefore, subtle phenomenon such as particle separation may be not inherent in the exact model. At least this is not clear in advance. Thus, to understand the dynamics of time-dependent stadium it is necessary to consider the moving boundary, i.e. to study the exact model (EM) in which the boundary motion is taken into account.

In the present paper we consider time-dependent stadium-like billiards and describe the results obtained for EM and for the model where the boundary displacement is neglected. The stadium-like system is remarkable by the fact that, in contrast to the Lorentz gas, chaos appears by the defocusing mechanism [27]. Also, in such systems an anomalous transport is observed [28]. We show that if the billiard geometry is close to the Bunimovich stadium, then in both cases Fermi acceleration is observed. But if the curvature of the focusing components is small enough such that this billiard corresponds to a nearly integrable system, then oscillations of the boundary for both cases lead to the separation of particles by their velocities.

It should be noted however, that to compare correctly the simplified version and the exact approach it is necessary to make statistically reliable calculations using the corresponding billiard geometry. To be more definite, we used the billiard geometry for which the critical velocity is clearly observed.

The paper is organized as follows. To be logical, in the next section we consider a stadium-type billiard with the fixed boundary and describe the corresponding billiard map obtained in Ref. [16]. To compare carefully the FBA and EM approaches, Section 3 presents generalizations of the known results related to the analysis of the FBA model [16,24]. In Section 4 we list the results of the thorough investigations of the EM and compare them with the results described in Section 3. Finally, in the conclusion we summarize our results.

2. Stadium with a fixed boundary

Assume that focusing components are arcs of radius R with the angular measure 2Φ (Fig. 1(a)) symmetric with respect to the vertical axis of the billiard. Thus, $R = (a^2 + 4b^2)/8b$, $\Phi = \arcsin(a/2R)$. In such a billiard chaos appears if the arc completing the focusing component to the entire circle belongs to the billiard table [27]. For $b \ll a$ this yields: $l/2R \approx 4bl/a^2 > 1$.

Analysis of the billiard dynamics is usually based on the unfolding method, i.e. reflecting the billiard table across its sides (see, e.g. Ref. [29]). To derive the corresponding billiard maps we use our approach described in Refs. [16,24].

Introduce dynamical variables as shown in Fig. 1(a). If the boundary is fixed, the incidence angle α_n^* is equal to the reflection angle α_n. Let V_n be the particle velocity and t_n the time of the nth collision with the boundary. To construct a map describing the dynamics of the particle in such a billiard, two cases must be considered:

(i) after a collision with the focusing component the billiard particle collides with this component again;
(ii) the next collision occurs with another focusing component.

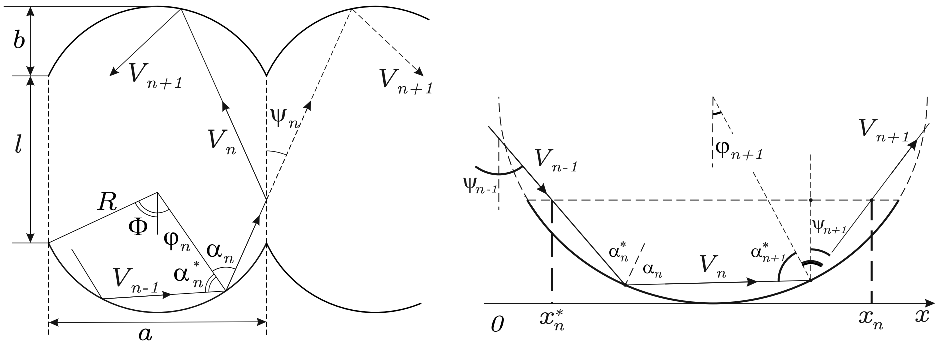

Fig. 1. Dynamical variables in stadium-like billiards.

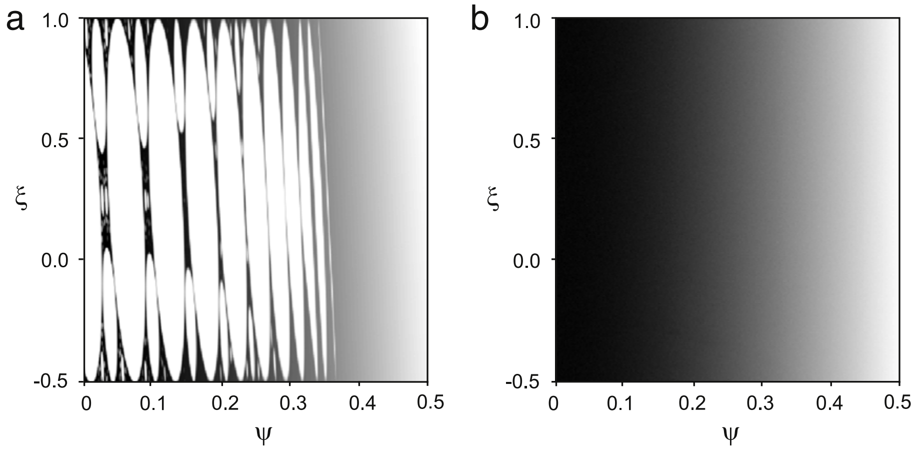

Fig. 2. Phase portrait of a stadium given by maps (1), (2). (a)—nearly integrable system, $a = 1$, $b = 0.01$, $l = 5$. (b)—developed chaos, $a = 0.5$, $b = 0.2$, $l = 1$.

In the former case, a simple geometric analysis (Fig. 1(b)) leads to the following map [16]:

$$\begin{aligned}
\alpha^*_{n+1} &= \alpha_n, \\
\alpha_{n+1} &= \alpha^*_{n+1}, \\
\varphi_{n+1} &= \varphi_n + \pi - 2\alpha_n \pmod{2\pi}, \\
t_{n+1} &= t_n + \frac{2R\cos\alpha_n}{V_n}.
\end{aligned} \tag{1}$$

If $|\varphi_{n+1}| < \Phi$, the particle makes a cascade of collisions with the same component. Otherwise, the $(n+1)$th collision occurs with the other focusing component. Then the map can be written as [16]:

$$\begin{aligned}
\alpha^*_{n+1} &= \arcsin\left[\sin(\psi_n + \Phi) - \frac{x^*_{n+1}}{R}\cos\psi_n\right], \\
\alpha_{n+1} &= \alpha^*_{n+1}, \\
\varphi_{n+1} &= \psi_n - \alpha^*_{n+1}, \\
t_{n+1} &= t_n + \frac{R(\cos\varphi_n + \cos\varphi_{n+1} - 2\cos\Phi) + l}{V_n\cos\psi_n},
\end{aligned} \tag{2}$$

where $\psi_n = \alpha_n - \varphi_n$, $x_n = [\sin\alpha_n + \sin(\Phi - \psi_n)]R/\cos\psi_n$ and $x^*_{n+1} = x_n + l\tan\psi_n \pmod{a}$.

Fig. 2 presents a phase portrait of a stadium in the $(\psi, \xi) = (\psi, \xi_n = 1/2 + (R\sin\varphi_n)/a)$ coordinate space given by maps (1), (2). The gray scale represents the number of points in the given region of phase space. One can see that in the nearly integrable case (Fig. 2(a)) stable fixed points are surrounded by invariant curves. The dynamics of particles in the neighborhood of these points is regular. The regions corresponding to different resonances are divided by separatrices surrounded by stochastic layers. The width of these layers is determined by the degree of nonlinearity of the system. As the nonlinearity increases, the fixed points lose their stability; as a result, a global chaos appears. This phenomenon occurs if $4bl/a^2 > 1$ (Fig. 2(b)).

Analyzing the obtained billiard map one can come to the conclusion [16] that the phase space dynamics of the billiard system in the neighborhood of stable fixed points is described by a twist map with the rotation number $\rho = \arccos\left(1 - 8bl/a^2\cos^2\psi_s\right)$, where $\psi_s = \arctan(sa/l)$, $s \in Z$, and $\{\xi = 1/2, \psi_s\}$ is one of a family of fixed points. Therewith the time between successive collisions of the particle with the boundary is $\tau \approx l/(V\cos\psi_s)$, where V is the particle velocity. Therefore, the rotation period is

$$T_{\mathrm{rot}} = \frac{2\pi}{\rho}\tau = \frac{2\pi l}{\arccos\left(1 - 8bl/(a\cos\psi_s)^2\right)\cos\psi_s}\frac{1}{V}, \tag{3}$$

where s corresponds to the number of fixed points, $s^2 \leq l/4b - l^2/a^2$ (see Ref. [16]).

3. The fixed boundary approximation

In this section, to compare two methods (EM and FBA), we analyze in detail a family of maps which describes the time-dependent stadium-type billiard in the fixed boundary approximation. Mainly we follow an approach proposed in paper [16].

Assume that the focusing components of the billiard are perturbed in such a way that their velocity at each point is the same in magnitude and directed normally to them. Let us suppose that the boundary oscillates according to a periodic law: $R = R_0 + r_0 f(\omega t + \eta)$, where ω is the frequency, and η is the phase. Thus, the boundary velocity is $U(t) = \dot{R}$.

When the particle velocity is high, $V \gg U$, one can obtain [16] that trajectories in the vicinity of fixed points should move along spirals. At that, the rotation period (3) remains almost the same as for the billiards with unperturbed boundary. Therefore, if V is equal to

$$V_r = \frac{\omega l}{\cos\psi_s \arccos(1 - 8bl/(a\cos\psi_s)^2)}, \tag{4}$$

then one can observe a resonance between the frequency of rotation around a fixed point and the frequency of the boundary oscillations.

Assume that the displacement of the boundary is small enough, i.e. $r_0 \ll b$. Then the map corresponding to such a billiard system can be written as [16]:

$$\begin{aligned} V_n &= \sqrt{V_{n-1}^2 + 4V_{n-1}\cos\alpha_n^* U_n + 4U_n^2}, \\ \alpha_n &= \arcsin\left(\frac{V_{n-1}}{V_n}\sin\alpha_n^*\right), \end{aligned} \tag{5}$$

$$\left.\begin{aligned} \alpha_{n+1}^* &= \alpha_n, \\ \varphi_{n+1} &= \varphi_n + \pi - 2\alpha_n \pmod{2\pi}, \\ t_{n+1} &= t_n + \frac{2R\cos\alpha_n}{V_n}, \end{aligned}\right\} \quad \text{if } |\varphi_{n+1}| \leq \Phi \tag{6}$$

$$\left.\begin{aligned} \psi_n &= \alpha_n - \varphi_n, \\ x_n &= \frac{R}{\cos\psi_n}\left[\sin\alpha_n + \sin(\Phi - \psi_n)\right], \\ x_{n+1}^* &= x_n + l\tan\psi_n \pmod{a}, \\ \alpha_{n+1}^* &= \arcsin\left[\sin(\psi_n + \Phi) - \frac{x_{n+1}^*}{R}\cos\psi_n\right], \\ \varphi_{n+1} &= \psi_n - \alpha_{n+1}^*, \\ t_{n+1} &= t_n + \frac{R(\cos\varphi_n + \cos\varphi_{n+1} - 2\cos\Phi) + l}{V_n\cos\psi_n}, \end{aligned}\right\} \tag{7}$$

if $|\varphi_n + \pi - 2\alpha_n| > \Phi$. Thus, this map presents the FBA model of the stadium. Expressions (6) correspond to a series of successive collisions of the particle with one focusing component and expressions (7) conform to a transition from one component of the boundary to another.

Following the approach developed in Ref. [16], consider map (5)–(7) in two qualitatively different cases: as a completely chaotic system and a nearly integrable model. In the first case, geometrically the billiard is close to a classical stadium: $\Phi \sim \pi/2$ (see Fig. 1). For further analysis the following billiard configuration has been chosen: $a = 0.5$, $b = 0.2$, $l = 1$, $r_0 = 0.01$, $\omega = 1$ and $V_0 = 0.2$. The corresponding phase portrait is a global stochastic sea (Fig. 2(b)). The particle velocity was calculated as the average over an ensemble of 5000 trajectories during 10^6 iterations with randomly chosen initial conditions (see Fig. 3, the central curve). As follows from the performed analysis, the particle velocity grows, and it has a complex dependence on n. Asymptotically, this is a power law: $V(n) \simeq n^\gamma$, $\gamma = 0.44$. In Fig. 4 this dependence (line 1) and the particle velocity (curve 2) are shown in the log–log plot. One can see that after 10^4 collisions these curves are practically coincided.

In Fig. 3 the maximal (the upper broken curve) and the minimal (the line practically coincided with the x-axis) velocities in the ensemble are also shown. The minimal velocity is not increasing, and it fluctuates within the range $V_{\min}^{\mathrm{FBA}} \in [6 \cdot 10^{-6}, 2 \cdot 10^{-3}]$. The maximal velocity, reached at the 10^6th collision, is $V_{\max}^{\mathrm{FBA}} = 53.65$. Therefore, the average velocity grows, and at $n = 10^6$ it runs the value $V = 5.6$.

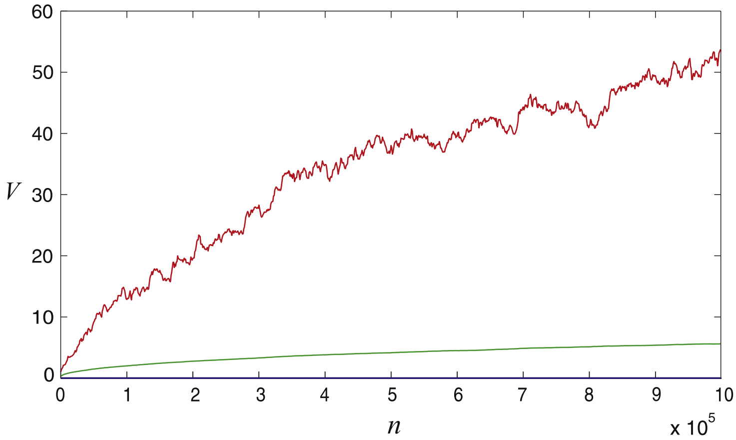

Fig. 3. (Color online) Average (green), highest (red), and smallest (blue) velocities of the ensemble of 5000 particles in the FBA billiard model (5)–(7) possessing the global chaos. $l = 1$, $a = 0.5$, $b = 0.2$, $r_0 = 0.01$, $\omega = 1$ and $V_0 = 0.2$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

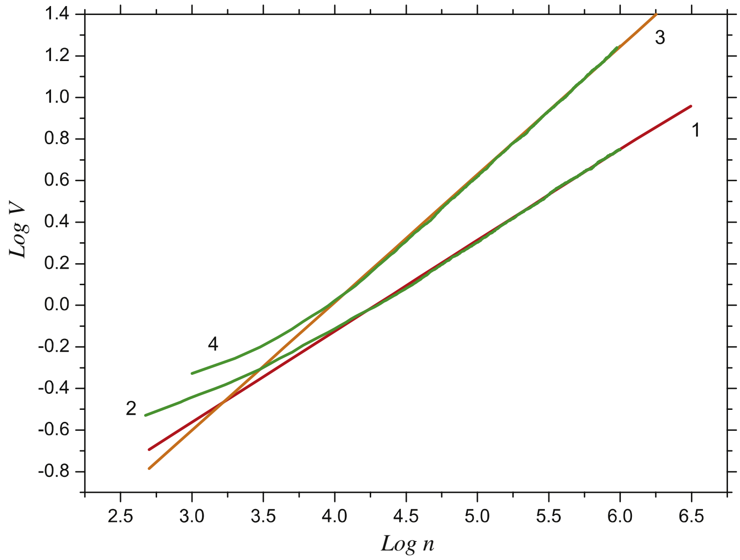

Fig. 4. (Color online) The growth of the velocity of the particle ensemble (curve 2) and its fitting (line 1) for FBA and EM (curve 4 and line 3).

Let us now turn to a nearly integrable billiard system. In this case the inequality $b \ll a$ hods true, so that the curvature of the focusing component brings only a weak nonlinearity in the system. For numerical investigations the following billiard parameters have been chosen: $a = 1$, $b = 0.01$, $l = 5$, $r_0 = 0.01$, $\omega = 1$. In such a configuration, the phase space has regions with regular and chaotic dynamics (see Fig. 2(a)). Appearance of such regions leads to the resonance, which occurs at the particle velocity value $V = V_r \simeq 5.38$ (see expression (4)).

Numerical analysis revealed that the dependence of the particle velocity on the number of collisions is different for the two sides of the resonance. If the initial velocity $V_0 < V_r$, the particle velocity V decreases up to a small enough value $V_{\min} < V_r$, and the velocity distribution of particles in the interval $(0, V_{\min})$ tends to a stationary one. But if $V_0 > V_r$, the particle velocity can reach high values. In this case, the distribution of particles is not stationary and unbounded. It is obvious that the value $V_{\min}$ depends on the billiard geometry (i.e. parameters a, b, l).

Fig. 5 presents the change of the average velocity of an ensemble of 1000 particles during 10^7 collisions with the boundary. Particle trajectories in this ensemble were different from each other in the initial values chosen at random in the focusing component and from a stochastic region. For every initial velocity V_0 three curves corresponding to minimal (curves 1 and 4), average (curves 2 and 5) and maximal (curves 3 and 6) velocity values in the ensemble were plotted. In Fig. 5 the dependence $V(n)$ for $V_0 = 5$ and $V_0 = 15$ is shown.

As follows from this dependence, at low initial velocities, $V_0 < V_r$, the minimal velocity (curve 1) goes down rapidly up to almost zero value, $V_{\min} \sim 10^{-6}$. Furthermore, the maximum particle velocity (curve 3) first grows slightly, but then it also reduces up to some value and fluctuates around it. Thus, the average velocity (curve 2) gradually decreases and approaches a constant value $\langle V \rangle \simeq 0.3$.

However, if initial velocities exceed some value, $V_0 > V_r$, then we have absolutely different phenomenon. The minimal particle velocity (curve 4) first decreases slowly, but next, when it overcomes the value V_r, this velocity immediately reduces

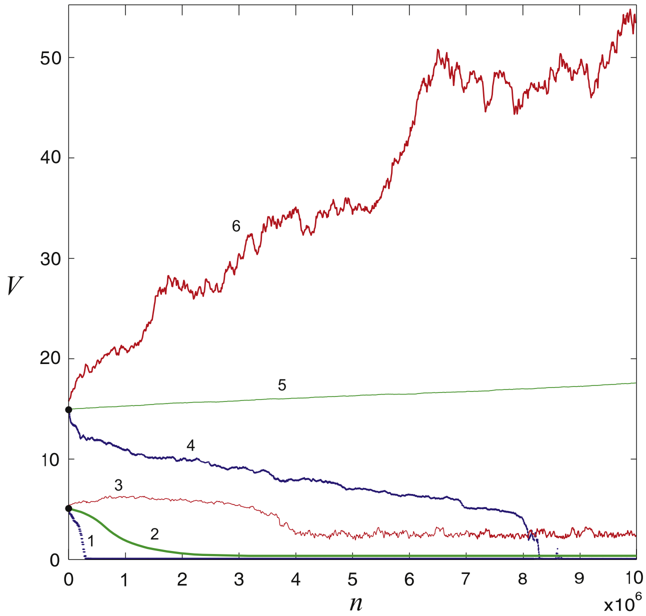

Fig. 5. (Color online) Dependence of the average velocity of the ensemble of 1000 particles on the number of collisions for a nearly integrable billiard system in FBA. $l = 5$, $a = 1$, $b = 0.01$, $r_0 = 0.01$, $\omega = 1$. $V_0 = 5$ and $V_0 = 15$, $V_r = 5.38$.

to a very low value $\sim 10^{-6}$ and oscillates around it. At the same time, the maximum velocity (curve 6) grows indefinitely. As a result, the average velocity (curve 5) increases, i.e. the Fermi acceleration phenomenon is inherent in such a case.

Hence, in this billiard system there exists a certain critical velocity V_c. Our numerical simulations (we checked more than 10 000 combinations of the billiard parameters) show that the critical velocity V_c coincides with the resonance value V_r given by expression (4). Therefore, the mechanism, which leads to the separation of fast and slow particles, arises from the difference in the particle dynamics when $V < V_r$ or $V > V_r$.

4. The exact model

In this section we numerically analyze the EM, in which the boundary displacement is taken into account. Such an approach has been already applied in general [30], to elliptic billiards [31], to the Fermi–Ulam system [10] and others (see Ref. [18] and the references cited therein).

To correctly investigate the EM we should take into account that the condition $a \le 2 \max R(t)$ should be held true. Also, the change of the radii of focusing components leads to variations of the length l. To simulate numerically such a system, instead of maps (5)–(6) it is more reliably and simple to use the method of molecular dynamics. However, to optimize our approach we used Eq. (7), where $R = R_0$, which makes it possible to estimate the transition time from one focusing component to another. Then the particle dynamics has been explicitly calculated during the time interval $\Delta t = \min\{r_0/V_n, 2\pi/\omega\}/N$, where $N \gg 1$. In the final stage the collision time has been found.

To compare the obtained results with the FBA model, the same billiard configuration $l = 1$, $a = 0.5$, $b = 0.2$, $r_0 = 0.01$, $\omega = 1$, $V_0 = 0.2$ and the ensemble of 5000 particles during 10^6 collisions with the boundary have been considered. The particle velocity as a function of the collision number is shown in Fig. 6. The central curve corresponds to the average velocity value. The upper broken line corresponds to the higher particle velocity in the ensemble, the lower line corresponds to its minimal velocity.

Although up to the 10^6th collision the maximal velocity has a lesser value $V_{\max}^{\mathrm{EM}} = 50.6$ than for the FBM case ($V_{\max}^{\mathrm{FBA}} = 53.65$), the minimal velocity increases with n and by the 10^6th collision reaches the value $V_{\min}^{\mathrm{EM}} = 2.1$. As a result, the average velocity grows and runs the value $V = 17.9$. One can see that the acceleration in the EM case more than three times exceeds the acceleration obtained on the basis of FBA. Fig. 4 clarifies the quantitative difference FBA from EM for the stadium with the fully developed chaos. In this figure, curves 2 and 4 correspond to the ensemble velocity, lines 1 and 3 go with their fitting: $V(n) \simeq n^{\gamma}$. For EM $\gamma = 0.62$, but for FBA $\gamma = 0.44$.

Let us now turn to a nearly integrable case, $b \ll a$. To compare it with the FBA model, the same billiard parameters, $l = 5$, $a = 1$, $b = 0.01$, $r_0 = 0.01$, $\omega = 1$, and an ensemble of 1000 particles have been considered. For this geometry the resonance velocity is $V_r = 5.38$. Particle trajectories were different from each other by randomly chosen initial conditions from stochastic layers. As before, for each initial value three curves, characterizing the particle ensemble, were plotted (see Fig. 7). Curves 1 and 4 correspond to the minimal particle velocity in the ensemble, curves 3 and 6 corresponds to their maximal values, and curves 2 and 5—the averaged velocities.

As follows from the obtained results, for low initial values, $V_0 < V_r$, there are particles in the ensemble the velocity of which (curve 1) goes down to a small value $V_{\min} \sim 10^{-2}$, and highly fluctuate. The mean amplitude of such fluctuations

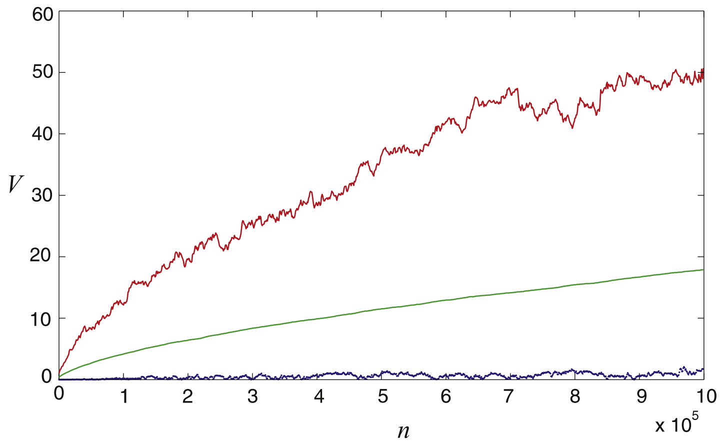

Fig. 6. (Color online) The same dependence and parameters as in Fig. 3, but for the EM.

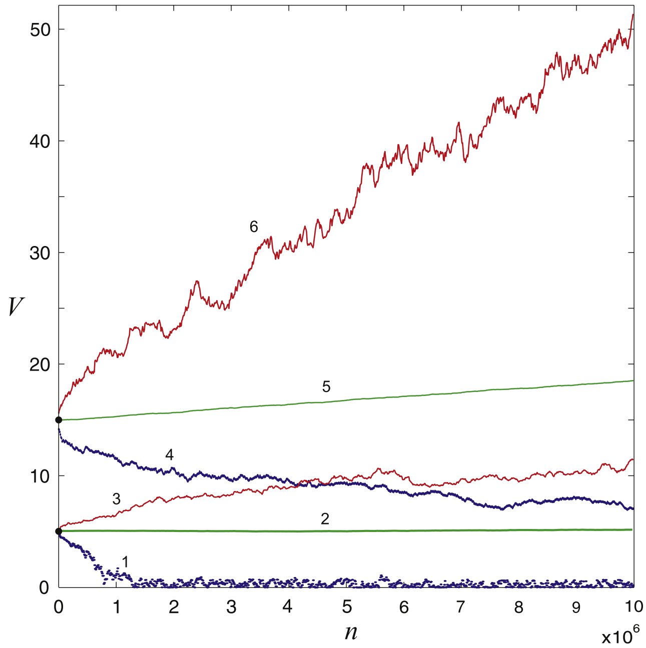

Fig. 7. (Color online) The same dependence and the billiard parameters as in Fig. 5, but for the EM.

is ∼0.6. At the same time, there exist particles with slowly increasing velocities (curve 3). However the fraction of such particles in the ensemble is relatively small. As a result, the averaged velocity does not grow and remains constant during whole time of the observation.

For high initial velocities, $V_0 > V_r$, (curves 4–6) the qualitatively different behavior is observed. The minimal velocity, as for the previous case, decreases (curve 4). However, the maximal velocity is rapidly increasing infinitely (curve 6). Therefore, the averaged velocity also increases (curve 5), i.e. Fermi acceleration is observed.

The deep analysis of the obtained results (we analyzed more than 1000 combinations of billiard parameters) allows us to clarify the particle separation phenomenon [32]. The resonance velocity for the chosen parameters is $V_r = 5.38$. If the initial velocity is close to it ($V_0 = 5$, see Fig. 7), in the phase space this corresponds to the case when particles are mainly in the neighborhood of the stability regions. If, however, the initial velocity of particles is low, then they are distributed sufficiently rapidly in the interval $[0, V_r]$. In principle, particles may penetrate into the region of high velocities, but only after a quite long period of time. However, the fraction of such particles is small enough. If the initial velocity exceeds V_r, then the spread (scatter) of the velocity values grows more rapidly in contrast to the case when they have a low velocity.

Thus, the main idea is the following. Near the stability regions the billiard particles cannot change their velocity, but the stability regions are accessible when the initial particle velocity is close to the resonance value.

5. Conclusion

To simulate the dynamics of time-dependent billiards one of the three basic methods is usually used: the fixed boundary approximation (FBA), the hopping approximation, and the exact model (EM). Each of these methods is a quite effective way to understand the dynamics of billiard particles. However, some approximations can lead to systematic underestimations of certain phenomena and even to incorrect results.

In the present paper, on the basis of stadium-like billiards with the periodically perturbed boundary, we have compared the numerically obtained results for the FBA and the EM case. As follows from our analysis, for the developed chaos in the billiard, the dependence of the particle velocity on the number of collisions has a power character (Fig. 4). Thus, in both cases Fermi acceleration is observed. However, the EM yields a value which is more than three times exceeds the velocity in the billiard investigated in the FBA.

At the same time, for a nearly rectangle stadium the particle separation is observed. In the FBA model, depending on the initial values, the particle ensemble can be accelerated or moderated. The EM demonstrates the same phenomenon in the sense that the particle separation is observed as well. However, the existence of critical velocity turns out to be less evident. In other words, for EM the separation phenomenon is a sufficiently subtle effect. Therefore, for the nearly integrable case the behavior of the billiard particles in FBA and in EM is qualitatively similar. This similarity can be probably explained by the existence of resonance velocity in the system. Although we present only one fixed set of billiard parameters for which the particle separation is observed, this phenomenon is observed for many other parameter values in FBA and in EM.

The effect of the separation of billiard particles can be treated as a specific Maxwell's demon, when by means of external perturbations of the billiard boundary it becomes possible to separate particles in the ensemble by their velocities. For the FBA model, the detailed exploration of this phenomenon recently was described in paper [24]. The EM case remains to be carefully analyzed [32].

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