Control of dynamical systems behavior by parametric perturbations: An analytic approach

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The problem of parametric suppression of deterministic chaos is considered. It is proved that certain parametric perturbations of a one-dimensional map with chaotic dynamics can lead to a transition of that map into a regime of regular behavior.

I. INTRODUCTION

In previous papers the problem of suppression of deterministic chaos in dynamical systems has been considered quite often (see, for example, Alekseev and Loskutov,[1(a)] Alekseev and Loskutov,[1(b)] Corbet,[2] Gribkov and Kuznetsov,[3] Lima and Pettini,[4] Plapp and Hübler,[5] Neymark and Landa,[6] and Rajasekar and Lakshmanan[7]). This is due to the fact that in this field one can get a series of interesting results concerning, on the one hand, problems in new artificial intelligent systems creation (Basti *et al.*[8] and Loskutov and Tereshko[9(a)]), and, on the other hand, problems of prediction of chaotic systems behavior when their parameters are slightly perturbed, and control of chaotic behavior (Alekseev and Loskutov,[10] Ott *et al.*,[11] Shinbrot *et al.*,[12] Chen and Dong,[13] and Dressler and Nitsche[14]).

The phenomenon of degeneration of chaotic dynamics into a periodic one as the result of purely parametric perturbations of a system with a strange attractor was numerically investigated by Alekseev and Loskutov,[1(a),1(b)] and was called the *phenomenon of parametric destochastization*. In the present study we consider this phenomenon analytically and more accurately. We prove on the example of the logistic map that such perturbations really can lead to a suppression of deterministic chaos and appearance of periodic dynamics. [Note, that the term "stochastic" is often used interchangeably with the term "chaotic" to signify deterministic chaos (Vul *et al.*[15] and Lichtenberg and Lieberman[16]). That is the reason why sometimes we also use the word "destochastization" meaning "the suppression of deterministic chaos."]

One-dimensional maps are a convenient object for investigations of nonlinear phenomena. They allowed us to understand many common properties which are inherent in real systems (Collet and Eckmann,[17] Preston,[18] Schuster,[19] and Sharkovskii *et al.*[20]). On the other hand, one-dimensional maps are more easy to study than systems of differential equations. So, consider a nonlinear one-dimensional map:

$$x_{n+1}=f(x_n,a), \tag{1}$$

where f is some function and a is a control parameter. The dynamics of the map (1), depending on the initial condition x_0 and values of the parameter a, may be both regular and chaotic. (Definitions of these terms are given below.) Denote the set of values of the parameter a leading to chaotic dynamics of the map (1) by A_c. Thus if $a\in A_c$, then the map (1) will generate for certain x_0, full aperiodic sequences of points not tending with an increase in n to any periodic orbit.

Assume that in the map (1) the control parameter a can vary with each iteration: $a=a_n$. This variation may be periodic or aperiodic. In the first case the sequence of values of the parameter a_n will consist of identical subsequences of a certain length, for example T: $a_k\neq a_m$, $m,k<T$, $m\neq k$, $a_{n+T}=a_n$, $a_{n+k}\neq a_n$ for $1\leqslant k<T$. Suppose now that variation of the parameter a happens in the set A_c, i.e., $a_n\in A_c$ for any n. Then one can show that in the set A_c there exists a certain subset (which we shall denote by A_d, for the word "*destochastization*") such that map (1) with $a=a_n\in A_d\subset A_c$ will possess periodic behavior. In other words, with periodic perturbations of the control parameter a of map (1), destochastization takes place. The rigorous consideration of this phenomenon is given below.

II. PARAMETRIC CHAOS SUPPRESSION

Let us consider a concrete, well-known, one-dimensional map—the logistic map. This map is a deeply understood dynamical system which can exhibit the chaotic property. Usually the logistic map is written as follows: $f_a:I\rightarrow I$, $f_a=ax(1-x)$, where a is the control parameter; f_a exists for all $x\in\mathbb{R}^1$. If $a\in[0,4]$ then $f_a(0)=f_a(1)=0$, $\max[f_a(x)]=f_a(1/2)=a/4$, and an action of the map f_a turns the interval $[0,1]$ into itself. In this case the logistic map can be written in the form:

$$\begin{aligned}&F_a:\ [0,1]\rightarrow[0,1],\\&F_a=ax(1-x),\quad a\in[0,4].\end{aligned} \tag{2}$$

We consider only this case. Depending on values of the parameter a map (2) can have two different types of behavior: regular and chaotic. The *first* type corresponds to such values of the parameter a at which almost all (in the sense of the Lebesgue measure) trajectories of the map F_a converge to a stable periodic orbit. The *second* type is determined by values of the parameter a at which behavior of the map F_a is chaotic.

At the moment there are several different definitions of the chaotic behavior of maps which apparently do not reduce to each other. The abstract definition belongs to Vul *et al.*[15] However, we shall use a more appropriate (from our point of view) definition (see also Takens[21]) that is based on the concepts of topological transitivity and sensitive dependence on initial conditions (see Eckmann and Ruelle,[22] Devaney,[23] Guckenheimer and Holmes,[24] and Wiggins[25]).

Let g be a map on an interval I.

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Definition. Let J be a compact invariant set for the map $g:I\to I$. Then g is called *chaotic* if the following points hold:

(a) g has sensitive dependence on initial conditions on J;

(b) g is topologically transitive on J.

Following the theorems of Ognev[26] and Misiurewicz[27] a unimodal one-dimensional map with negative Schwarzian derivative has chaotic behavior if a trajectory of its critical point coincides with an unstable periodic orbit of a finite period, beginning with some finite step of iterations. Therefore, if the point $x_c=1/2$ of the logistic map (2) comes to an unstable finite-periodic orbit (or an unstable fixed point) then this map will be chaotic. Denote a set of values of the parameter a, corresponding to the chaotic behavior of F_a by A_c. The Lebesgue measure of this set is positive, $\mu_L(A_c)>0$ (Jakobson[28,29]).

Let us suppose now that a parametric perturbation with period T acts on the map F_a, $a\in A_c$, from iteration to iteration. It means that in the set A_c a map of transformations G: $A_c\to A_c$ works. Then the map (2) can be written as follows:

$$F_a:\ x\to ax(1-x),\qquad G:\ a\to g(a),\quad a\in A_c \tag{3}$$

where $a_{i+1}=g(a_i)$, $a_i\neq a_j$, $i,j<T$, $i\neq j$, and $a_1=g(a_T)$. Let us introduce the notations for the case $T=2$: $g(a)\equiv a_1$, $g(a_1)=g^{(2)}(a)\equiv a_2$. Then the following theorem holds.

Theorem. There exists a set A_d which consists of the pairs a_1^d, $a_2^d\in A_d$ such that the map (3) with $T=2$, $a_1=a_1^d$, $a_2=a_2^d$ generates stable cycles of finite period.

The major meaning of this theorem consists of the fact that certain periodic parametric perturbations of the chaotic logistic map turn it from the family of the maps of the second type into the family of the first one (see above). It is obvious that for the proof it is sufficient to find only one pair a_1^d, a_2^d of parameters a_1, a_2 from the set A_c, which obeys the condition of the theorem.

Proof. Taking into account the above notations, the map (3) will have the following explicit form:

$$x_{2n+1}=f_1(x_{2n},a_1)=a_1x_{2n}(1-x_{2n}),\qquad x_{2n+2}=f_2(x_{2n+1},a_2)=a_2x_{2n+1}(1-x_{2n+1}). \tag{4}$$

Let us introduce the following functions: $\Phi_1(x,a_1,a_2)=f_1(f_2)$, $\Phi_2(x,a_1,a_2)=f_2(f_1)$. It is not hard to see that Φ_1 generates the odd iterations, and Φ_2 generates the even ones. That is why, determining the initial value x_0 and the first iteration, $x_1=f_1(x_0)$, the map (4) can be written as follows:

$$x_{2n+1}=\Phi_1(x_{2n-1},a_1,a_2), \tag{5a}$$

$$x_{2n+2}=\Phi_2(x_{2n},a_1,a_2). \tag{5b}$$

Each of the maps Φ_1 and Φ_2 executes transformations independently from each other for the given initial conditions x_0 and $x_1=f_1(x_0)$. Thus the map (4) consists of the two consecutive iterations (5a) and (5b). Therefore any of the cycles of period 2τ of map (4) simultaneously will be a "semicycle" of period τ of map (5a) and a "semicycle" of the same period τ of map (5b). Each of these latter semicycles of period τ will be the fixed points $\tilde{x}_i^1$, $i=1,2,\ldots,\tau$, of the map $\Phi_1^{(\tau)}$ and the fixed points $\tilde{x}_i^2$, $i=1,2,\ldots,\tau$, of the map $\Phi_2^{(\tau)}$. The inverse statement, generally speaking, is not necessarily true: not only the cycles of the period τ, but also the cycles of the periods $k=\tau/j$, $j=2,3,\ldots,\tau$, where k is integer, are fixed points of the maps $\Phi_1^{(\tau)}$ and $\Phi_2^{(\tau)}$. But because the map with chaotic behavior possesses only *unstable* cycles, the existence of the *stable* fixed points $\tilde{x}_i^1$ and $\tilde{x}_i^2$, $i=1,2,\ldots$, of the maps $\Phi_1^{(\tau)}$ and $\Phi_2^{(\tau)}$, respectively, means immediately that the map (5) and consequently the map (4) has regular dynamics.

However, the stable fixed points $\tilde{x}_i^1$, $\tilde{x}_i^2$ can form not only one, but several stable cycles. In this case the periodic behavior of map (4) will be provided by a stable cycle of one of the periods $k=2\tau/j$, $j=1,2,\ldots,\tau$, depending on the initial value. In order to distinguish which points form one or another stable cycle, for the chosen number τ it is necessary to consider for each $i=1,2,\ldots,\tau-1$ the following relations of polynomials:

$$[\Phi_1^{(\tau)}(\tilde{x}^1,a_1,a_2)-\tilde{x}^1]/[\Phi_1^{(i)}(\tilde{x}^1,a_1,a_2)-\tilde{x}^1],\qquad [\Phi_2^{(\tau)}(\tilde{x}^2,a_1,a_2)-\tilde{x}^2]/[\Phi_2^{(i)}(\tilde{x}^2,a_1,a_2)-\tilde{x}^2]. \tag{6}$$

Then the common points $\tilde{x}_i^1$, $\tilde{x}_i^2$ for all $i=1,2,\ldots,\tau-1$ [this corresponds to the minimal number of zeros of (6)] converting the relations (6) into zero, will make a cycle of period 2τ in map (4). When the stable cycle is unique, then an arbitrary number $m>\tau$ (where τ is semiperiod of this cycle, which is multiple of m), the number of stable fixed points $\tilde{x}_i^1$, $\tilde{x}_i^2$ of the functions $\Phi_1^{(m)}$, $\Phi_2^{(m)}$, is always 2τ, i.e., $i=\tau$.

Thus it is necessary to find values $a_1=a_1^d\in A_c$, $a_2=a_2^d\in A_c$, at which the maps $\Phi_1^{(\tau)}$ and $\Phi_2^{(\tau)}$, $\tau<\infty$, have stable fixed points.

Let us consider maps (4) and (5) for the following parameter values: $a_1=3.678\,573\,36\ldots$, $a_2=3.974\,591\,25\ldots$. It is known (cf. Sharkovskii *et al.*[20]) that in the case $a=a_1$ the critical point x_c of the quadratic map (2) comes into the unstable fixed point after three iterations. At $a=a_2$ the point x_c comes into the unstable cycle of period two after four iterations. Consequently $a_1\in A_c$, $a_2\in A_c$. Now it is necessary to solve the following equations:

$$\phi_1^{(\tau)}(\tilde{x}^1,a_1,a_2)=\tilde{x}^1,\qquad \phi_2^{(\tau)}(\tilde{x}^2,a_1,a_2)=\tilde{x}^2. \tag{7}$$

From direct calculations we can see that at $\tau=3$ and the chosen parameters a_1,a_2 each of the maps $\Phi_1^{(\tau)}$ and $\Phi_2^{(\tau)}$ has three stable fixed points, respectively, $\tilde{x}_i^1$ and $\tilde{x}_i^2$, $i=1,2,3$. These points form the stable cycle of period 6 (see Fig. 1) in map (4). Therefore the given values a_1 and a_2 compose one of the pairs, $a_1=a_1^d$, $a_2=a_2^d$, of the set A_d and thus they satisfy the condition of the theorem.

It is necessary to note that the chosen values a_1,a_2 are *irrational*. Consequently, any numerical investigation of the functions $\Phi_1^{(\tau)}(\tilde{x}^1,a_1,a_2)$ and $\Phi_2^{(\tau)}(\tilde{x}^2,a_1,a_2)$ does not give the rigorous results about the stable fixed points $\tilde{x}_i^1$, $\tilde{x}_i^2$, $i=1,2,3$. However, it is not easy to see that these points appear as a typical case: their stability does not break as a result of small perturbations of the values a_1 and a_2 (see Fig. 2). Therefore, in the close vicinity of the chosen parameters a_1,a_2 (i.e., at the irrational ones as well) the result shown in

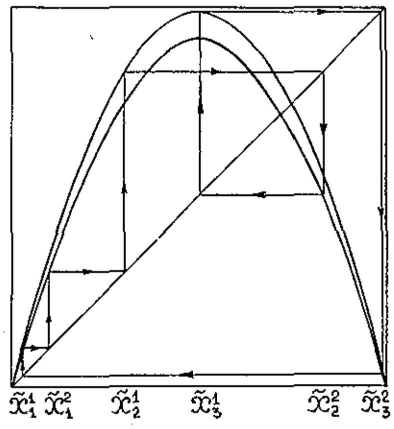

FIG. 1. The stable cycle of period 6 in map (3) at $a_1=3.678\,573\,36$, $a_2=3.974\,591\,25$ which is formed by the following stable fixed points: $\tilde{x}_1^1=0.023\,225\,32...$, $\tilde{x}_2^1=0.301\,779\,44...$, $\tilde{x}_3^1=0.500\,679\,08...$, $\tilde{x}_1^2=0.090\,167\,20...$, $\tilde{x}_2^2=0.837\,480\,58...$, $\tilde{x}_3^2=0.993\,645\,94...$.

the Fig. 1 will be qualitatively the same: in map (3) at two-periodic parametric perturbation the stabilization of chaotic dynamics of the quadratic maps family takes place. ■

Remark 1. The theorem can be extended if we consider the case $T>2$. Then calculations of T values $a_1^d, a_2^d,..., a_T^d$ become much more complicated. Therefore the proof for arbitrary T seems to be possible with the help of numerical investigations.

Remark 2. We have given the example of only one pair of the values a_1^d, a_2^d from the set A_d, satisfying the condition of the theorem. But in addition to the mentioned parameters there are many other pairs, a_1^d, a_2^d, in A_d which also obey the theorem. However, their location is a very difficult problem, which nevertheless is solved easily numerically.

Hypothesis. From the numerical investigations it follows that apparently the set A_d obeys the inequalities: $0<\mu_L(A_d)<\mu_L(A_c)$.

The case is interesting when we have chosen as a_1 the point of accumulation a_∞ of the logistic map, $a_1=a_\infty$, and we have taken as a_2 the value 4, $a_2=4$. The value a_∞ does not belong to the set A_c because at $a=a_\infty$ map (2) has no chaotic dynamics. The behavior of this map is infinitely periodic. For $a=4$ this map exhibits the most pronounced chaotic properties. Nevertheless, map (4) with the given values of parameters, $a_1=a_\infty$ and $a_2=4$, possesses a stable cycle of period four (Fig. 3).

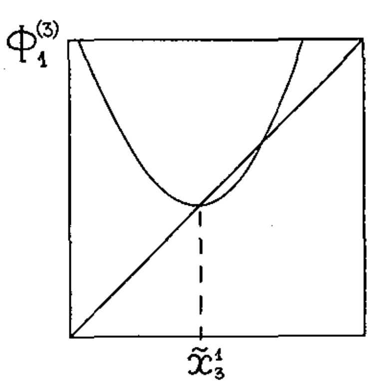

FIG. 2. One of the stable fixed points of the map $\Phi_1^{(3)}$.

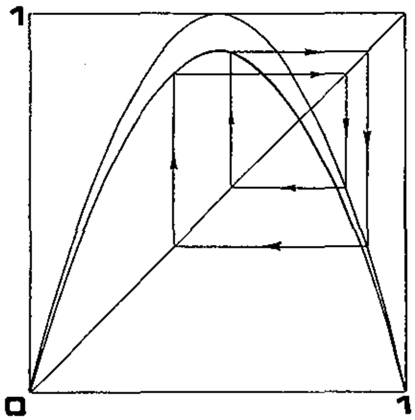

FIG. 3. The stable cycle of the period 4 of map (3) at $a_1=a_\infty$, $a_2=4$.

Thus with periodic parametric perturbation of the logistic map having chaotic behavior, parametric suppression of deterministic chaos takes place.

III. SUPPRESSION OF DETERMINISTIC CHAOS BY MEANS OF DETERMINISTIC CHAOS: IS IT POSSIBLE?

Up to the present we have investigated a problem concerning chaos suppression by means of an external periodic parametric action on the systems with chaotic dynamics. Here we shall look at a similar problem but with another position: Is there a possibility of suppressing chaos (in the rigorous sense) in a dynamical system by perturbing it with the action of another system also with chaotic dynamics? It is clear that this task as formulated is in a certain sense incredible and might be imagined intuitively impossible. Nevertheless, we can prove with the example of the same logistic map with chaotic dynamics that such a phenomenon is really possible. In fact it follows immediately from the statement of the theorem of Sec. II.

It should be noted that deep mathematical research (Blank[30]) and some numerical investigations (see, for example, Matsumoto and Tsuda,[31] Matsumoto,[32] Anishchenko and Safonova,[33] and Herzel[34]) concerned with regularization of chaotic dynamics by means of certain stochastic perturbations have been done. In contrast to these papers the major meaning of our considerations consists of the following: if there are two mappings with chaotic behavior which are different from each other by a value of parameters, then their composition does not necessarily possess chaotic dynamics. Namely, let f_1 be the logistic map F_a with one of several possible values of the parameter $a \equiv a_1 \in A_c$, and let f_2 be the same map, but with another value of the parameter $a \equiv a_2 \in A_c$, i.e., $f_1(a_1^i,x)=a_1^i x(1-x)$ and $f_2(a_1^i,x) = a_2^i x(1-x)$, $a_1^i \neq a_2^i$, a_1^i, $a_2^i \in A_c$, $i=1,2,...$. Thus f_1 and f_2 have chaotic dynamics. In this case the statement of the theorem of Sec. II leads to the following.

Corollary. The maps $g_1=f_1\circ f_2$ and $g_2=f_2\circ f_1$ for certain values of parameters $a_1^i=a_1^{di}$, $a_2^i=a_2^{di}$, a_1^{di}, $a_2^{di}\in A_c$, have stable periodic points.

Proof. The composition $f_1\circ f_2=g_1$ is the map Φ_1, $g_1=\Phi_1$, and the function $g_2=f_2\circ f_1$ is the map Φ_2, $g_2=\Phi_2$, which were described above. Therefore, all arguments concerning Φ_1 and Φ_2 should be repeated for g_1 and g_2. Further (see above), for $a_1=3.678\,573\,36...$ the map f_1 is chaotic and for $a_2=3.974\,591\,25...$ the map f_2 is chaotic. However, the functions Φ_1,Φ_2 and consequently the maps g_1,g_2 at the given values a_1,a_2 have stable period three points. ■

Remark. This corollary (as well as the theorem) may be extended for an arbitrary number of mappings $f_1,f_2,\dots,f_n$, $f_j=a_j^i x(1-x)$, $a_j^i\in A_c$, $i\neq j$, $i,j=1,2,\dots,n$. In this case the statement will be concerned with the compositions $g_1=f_1\circ f_2\circ\cdots\circ f_n$, $g_2=f_n\circ f_1\circ f_2\circ\cdots\circ f_{n-1},\dots$, $g_n=f_2\circ f_3\circ\cdots\circ f_n\circ f_1$.

Thus if a chaotic map acts on another chaotic map, then at certain conditions this interaction leads to a mutual chaos suppression.

IV. DISCUSSION

We have shown that chaos can be suppressed in the logistic map. Therefore, for systems in which the transition to chaotic dynamics happens through period doubling bifurcations and is described effectively by the logistic map, the phenomenon of parametric suppression of deterministic chaos can take place. Some time earlier we investigated the transition to chaos for the example of a circle map (Loskutov,[35] and Loskutov and Rybalko[36]). It was shown numerically that in this case the parametric suppression of chaos is observed as well. On the other hand, from the corollary it follows that certain interaction of systems with chaotic behavior can regularize their dynamics.

One can use these two phenomena for processing (sending, reading, and recording) latent information and for an explanation of certain brain behavior as an information processing system (Loskutov and Tereshko[9(a)]). It should be noted that the idea of sending secure signals has been achieved theoretically and experimentally (Pecora and Caroll,[37–39] and Gullicksen *et al.*[40]). Here some another approach is proposed.

So, assume that we have a chaotic dynamical system. Suppose that the chaotic behavior of this system is provided by a chaotic attractor which appears through period doubling bifurcations. When chaos takes place, all orbits with periods 2^nT, $n=0,1,\dots$, are unstable. Let every periodic orbit correspond to some "word" the length of which is equal to the period. For example, the words $\{A\}$, $\{AB\}$, $\{ABCD\}$, $\{ABCDEFGH\}$,... should be associated with the orbits of periods T, $2T$, $4T$, $8T$,..., respectively (it can be done a certain way). Since periodic orbits are unstable then all information (words) encoded in them will be hidden. Now, if we perturb parametrically such a system then at certain conditions we may stabilize certain orbits and thus extract the hidden information. Moreover, if this system will be coupled with another similar system, then in this way one can construct new models of ciphering and deciphering information and some device of the codal-lock type (Loskutov and Tereshko[9(b)]). In contrast to traditional ways of extracting information, in the given case the necessary condition is the following: the key for ciphering and deciphering must be dynamical. Besides, the described method ensures the easy transitions of the system between stabilized orbits, since the values a_1^{di}, a_2^{di-1}, $i=2,3,\dots$, may be very close.

Finally, although we have considered chaos suppression on the basis of localized systems only, in distributed systems which can be approximated by a two-dimensional lattice of coupled one-dimensional maps, chaos can be suppressed as well. Now we have preliminary numerical results (Loskutov and Thomas[41]).

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