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PARAMETRIC PERTURBATIONS AND DYNAMIC SYSTEM CONTROL

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The article analyzes dynamical systems with externally applied periodic perturbations in a general setting. We provide a rigorous justification of an approach that reduces such systems to autonomous systems and thus simplifies the analysis. The behavior of families of quadratic one-dimensional maps and circle maps in the presence of parametric perturbations is studied in detail. We prove the existence of periodic perturbations acting strictly on a chaotic subset that stabilize the dynamics and induce the emergence of stable cycles in initially chaotic maps. The analytical results are supplemented with numerical data. It is shown that chaos may be suppressed by a sufficiently complex periodic perturbation.

1. Introduction

Dynamical systems with chaotic behavior are a subject of intensive studies. This is primarily attributable to the fact that chaos is a fairly general property of diverse nonlinear processes observed in many natural sciences — from biology to chemical kinetics. The reason for chaotic behavior is not the complexity of the dynamical systems, but rather the action of external perturbations. Moreover, chaotic behavior is not necessarily observed in complex systems with an arbitrarily large number of interacting particles, while chaotic oscillations may develop in very simple systems with as few as one and a half degrees of freedom. The appearance of chaos is associated with purely internal features of the dynamical system, when its trajectories become exponentially unstable for certain parameter values.

This has led to the emergence of a new line of research in deterministic chaos theory focusing on control of chaotic systems and suppression of chaos (or stabilization of chaotic behavior). This line of research is related to many branches of physics and mathematics: alongside the main issues of control and predictability, the chaos suppression problem touches on a whole range of important applications such as information processing (i.e., recording, encoding, decoding, and hidden transmission of useful messages; see, e.g., [1–3]), self-organization [4, 5], stabilization of unordered contractions of the cardiac muscle and defibrillation [6–9], artificial creation of coherent structures in distributed systems with spatial-temporal chaos [10] and their approximation by interlinked map lattices [11], and others. Our ability to solve even a part of these problems would substantially improve the understanding of processes and regularities underlying the behavior of diverse nonlinear dynamical systems and significantly advance the development of nonlinear oscillation theory for both lumped and distributed systems.

By *stabilization* of instability or chaotic behavior of dynamical systems we usually understand artificial creation of stable (as a rule periodic) oscillations in the system by application of external multiplicative or additive perturbations. In other words, stabilization requires finding external perturbations that move the system from a chaotic to a regular regime. Despite this attractively simple wording, the problem is quite difficult to solve for a whole range of dynamical systems. Moreover, the solution of the stabilization problem is still far from completion, although an impressive number of publications deal with this topic (see, e.g., [12–14] and the lists of references in [15, 16]).

The chaotic behavior of dynamical systems can be stabilized in two different ways. The first approach moves the system from chaotic to regular regime by applying external perturbations without feedback. In other words, this approach ignores the current state of the dynamical system variables. A qualitatively different approach applies a correcting perturbation allowing for the target value of the dynamical variables and therefore involves feedback as an integral component of the dynamical system. According to established terminology, the first approach to chaotic dynamics stabilization is called *chaos suppression* or *feedback-less chaos control*, while the second approach is

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called *feedback chaos control* (or controlling chaos). Either method can be implemented using parametric or force techniques.

The first parametric method of chaos suppression (without feedback) is apparently described in [17]. It has been subsequently substantiated for specific examples of a certain class of dynamical systems [18]. Multiplicative perturbations have been assumed to remain always inside the region of chaotic behavior. Subsequently similar approaches have been described by many authors (see, e.g., [12, 19–21] and the references therein). A sufficiently general feedback-less force control method for chaotic systems has been proposed by Huebler's group [22]. Other methods for stabilization of chaotic dynamics have been considered in [13, 14, 23]. Interesting general rules for chaotic behavior stabilization by force control of dynamical systems have been described in [24], where it is assumed that the threshold of chaos suppression by an additive perturbation is related with system entropy by a scaling relationship.

Feedback methods have become very popular after the publication of the Maryland group study [25, 26], where it is shown that sufficiently weak parametric perturbations can be applied to stabilize virtually any saddle-point limiting cycle enclosed in a chaotic attractor. By correcting the parameters in accordance with the value of the dynamical variables, we can force the system to function on a preselected limiting cycle. The publication of [25] has stimulated both experimental and numerical studies of chaotic behavior stabilization (see the references in [15, 16]) and has attracted increased attention to controlling unstable systems.

The present article examines the effect of small periodic perturbations on one-dimensional maps. The article is organized as follows. First the general propositions are proved (Sec. 2); then we consider in detail the family of maps

$$x_{n+1} = ax_n(1 - x_n), \tag{1}$$

where $a \in (0;\, 4]$ and $x_n \in (0;\, 1)$, and

$$x_{n+1} = a + x_n + b\sin x_n, \qquad \mod 2\pi, \tag{2}$$

where $a > 0$ and $b > 1$. The behavior of these systems is studied in detail in the presence of parametric perturbations of period 2. The bifurcation diagrams obtained in the presence of this perturbation illustrate the harmony of chaos and order and are comparable in their beauty to some fractal sets generated by iteration of complex functions.

2. Methods of Analysis of Perturbed Maps

In this section we describe some general properties of maps in the presence of parametric perturbation. Assume that the map describing the behavior of some process has the form

$$T_a\colon\ \mathbf{x} \longmapsto \mathbf{f}(\mathbf{x},\, a), \tag{3}$$

where $\mathbf{x} \in M \subset \mathbb{R}^n$, a is a parameter from the set of admissible values $A \subset \mathbb{R}$, $\mathbf{x} = \{x_1,\, x_2,\, \ldots,\, x_n\}$, and $\mathbf{f} = \{f_1,\, f_2,\, \ldots,\, f_n\}$.

We define a parametric perturbation as a transformation that determines the value of a at each time instant, $G : A \to A$. Then the map (3) is representable in the form

$$\mathbf{T}_a\colon \begin{cases} \mathbf{x} \longmapsto \mathbf{f}(\mathbf{x},\, a), \\ a \longmapsto g(a). \end{cases} \tag{4}$$

A perturbation is called *periodic* of period τ (or τ-periodic) if the function $g(a)$ is defined only at τ points $a_1,\ a_2,\ \ldots,\ a_\tau$ in the following way: $a_{i+1} = g(a_i),\ i = 1, \ldots, \tau - 1$, and $a_1 = g(a_\tau)$. In other words, the perturbation is defined by τ parameters that are sequentially "switched" in map (3). The set of perturbations of period τ can be placed in correspondence to the set

$$\mathbf{A} = \left\{ \widehat{a} \in \underbrace{A \otimes A \otimes \ldots \otimes A}_{\tau} : \widehat{a} = (a_1,\, a_2,\, \ldots,\, a_\tau),\ a_i \neq a_j,\ 1 \le i,\, j \le \tau,\ i \neq j,\ a_1,\, a_2,\, \ldots,\, a_\tau \in A \right\},$$

$\mathbf{A} \subset \mathbb{R}^\tau$.

The introduction of a τ-cyclic perturbation for the map (3) implies that the resulting system (4) can be written as

$$\mathbf{T} = \begin{cases} T_{a_1}:\ \mathbf{x} \longmapsto \mathbf{f}(\mathbf{x},\, a_1) \equiv \mathbf{f}_1, \\ T_{a_2}:\ \mathbf{x} \longmapsto \mathbf{f}(\mathbf{x},\, a_2) \equiv \mathbf{f}_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ T_{a_\tau}:\ \mathbf{x} \longmapsto \mathbf{f}(\mathbf{x},\, a_\tau) \equiv \mathbf{f}_\tau. \end{cases} \tag{5}$$

We introduce τ functions of the following form:

$$\begin{aligned} &\mathbf{F}_1 = \mathbf{f}_\tau\big(\mathbf{f}_{\tau-1}(\ldots \mathbf{f}_2(\mathbf{f}_1(\mathbf{x}))\ldots)\big), \\ &\mathbf{F}_2 = \mathbf{f}_1\big(\mathbf{f}_\tau(\mathbf{f}_{\tau-1}(\ldots \mathbf{f}_3(\mathbf{f}_2(\mathbf{x}))\ldots))\big), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\mathbf{F}_\tau = \mathbf{f}_{\tau-1}\big(\mathbf{f}_{\tau-2}(\ldots \mathbf{f}_1(\mathbf{f}_\tau(\mathbf{x}))\ldots\big), \end{aligned} \tag{6}$$

where $\mathbf{x} = \{x_1,\, x_2,\, \ldots,\, x_n\}$ and

$$\mathbf{f}_i = \{f_i^{(1)},\, f_i^{(2)},\, \ldots,\, f_i^{(n)}\}, \qquad \mathbf{F}_i = \{F_i^{(1)},\, F_i^{(2)},\, \ldots,\, F_i^{(n)}\}, \qquad i = 1,\, 2,\, \ldots,\, \tau,$$

are n-component functions. Then the perturbed map (4) obviously can be rewritten as

$$\begin{aligned} &T_1:\ \mathbf{x} \longmapsto \mathbf{F}_1(\mathbf{x},\, a_1,\, a_2,\, \ldots,\, a_\tau), \\ &T_2:\ \mathbf{x} \longmapsto \mathbf{F}_2(\mathbf{x},\, a_1,\, a_2,\, \ldots, a_\tau), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &T_\tau:\ \mathbf{x} \longmapsto \mathbf{F}_\tau(\mathbf{x},\, a_1,\, a_2,\, \ldots,\, a_\tau) \end{aligned} \tag{7}$$

with the initial conditions $\mathbf{x}_1 = \mathbf{f}_1(\mathbf{x}_0),\ \mathbf{x}_2 = \mathbf{f}_2(\mathbf{x}_1),\ \ldots,\ \mathbf{x}_{\tau-1} = \mathbf{f}_{\tau-1}(\mathbf{x}_{\tau-2})$.

Two important propositions are easily proved for these maps.

Lemma 1 [27, 28]. *If the map $T_k,\ 1 \le k \le \tau$, has a cycle of period t and the functions $\mathbf{f}_k(\mathbf{x})$ are continuous, then the map $T_p,\ p = k + 1 \pmod{\tau}$, has a cycle of the same period t.*

Moreover:

(a) if the cycle of the map T_k *is stable, then the cycle of the map* T_p *is also stable;*

(b) if $\mathbf{f}_k$ *is a homeomorphism, then the maps* T_k *and* T_p *are topologically equivalent.*

Proof. Assume that $\mathbf{f}_k(\mathbf{x})$, $1 \le k \le \tau$, is a C^0-function and T_k has a cycle of period t. This means that there exists a point $\widetilde{\mathbf{x}}$ such that $\mathbf{F}_k^t(\widetilde{\mathbf{x}}) = \widetilde{\mathbf{x}}$, $\mathbf{F}_k^j(\widetilde{\mathbf{x}}) \neq \widetilde{\mathbf{x}}$, $1 \le j < t$. Consider an expression that follows directly from the definition of $\mathbf{F}_k$:

$$\mathbf{f}_k\big(\mathbf{F}_k(\mathbf{x})\big) = \mathbf{F}_p\big(\mathbf{f}_k(\mathbf{x})\big), \qquad p = k + 1 \pmod{\tau}. \tag{8}$$

Then it is easy to obtain that $\mathbf{f}_k(\mathbf{F}_k^n) = \mathbf{F}_p^n(\mathbf{f}_k)$ and therefore for the point $\widetilde{\mathbf{x}}$ and $n = t$ we have $\mathbf{F}_p^t\big(\mathbf{f}_k(\widetilde{\mathbf{x}})\big) = \mathbf{f}_k\big(\mathbf{F}_k^t(\widetilde{\mathbf{x}})\big) = \mathbf{f}_k(\widetilde{\mathbf{x}})$. Moreover, for $1 \le j < t$ we have the inequality $\mathbf{F}_p^j\big(\mathbf{f}_k(\widetilde{\mathbf{x}})\big) \neq \mathbf{f}_k(\widetilde{\mathbf{x}})$, because if $\mathbf{F}_p^j\big(\mathbf{f}_k(\widetilde{\mathbf{x}})\big) = \mathbf{f}_k(\widetilde{\mathbf{x}})$, then $\mathbf{F}_p^j\big(\mathbf{f}_k(\widetilde{\mathbf{x}})\big) = \mathbf{f}_k\big(\mathbf{F}_k^j(\widetilde{\mathbf{x}})\big) = \mathbf{f}_k(\widetilde{\mathbf{x}})$. However, since the functions $\mathbf{f}_i$, $i = 1, 2, \ldots, \tau$, are single-valued, we can write

$$\mathbf{f}_{k-1}\big(\mathbf{f}_{k-2}(\ldots \mathbf{f}_k(\mathbf{F}_k^j(\widetilde{\mathbf{x}})))\big) = \mathbf{f}_{k-1}\big(\mathbf{f}_{k-2}(\ldots \mathbf{f}_k(\widetilde{\mathbf{x}}))\big)$$

(see (6)), i.e., $\mathbf{F}_k^{j+1}(\widetilde{\mathbf{x}}) = \mathbf{F}_k(\widetilde{\mathbf{x}})$. But this contradicts our assumption. In other words the point $\mathbf{f}_k(\widetilde{\mathbf{x}})$ is t-periodic for the map T_p.

If the point $\widetilde{\mathbf{x}}$ is a stable periodic point of the map T_k, then there exists a neighborhood $U \ni \widetilde{\mathbf{x}}$ such that for every point $\mathbf{x} \in U$ we have $\lim\limits_{n\to\infty} \mathbf{F}_k^{tn}(\mathbf{x}) = \widetilde{\mathbf{x}}$. Since the functions $\mathbf{f}_k$ are continuous, this implies that $\lim\limits_{n\to\infty} \mathbf{f}_k\big(\mathbf{F}_k^{tn}(\mathbf{x})\big) = \lim\limits_{n\to\infty} \mathbf{F}_p^{tn}\big(\mathbf{f}_k(\mathbf{x})\big) = \mathbf{f}_k(\widetilde{\mathbf{x}})$. In other words, all the points from the neighborhood $\mathbf{f}_k(U)$ are attracted to the point $\mathbf{f}_k(\widetilde{\mathbf{x}})$ under the action of the map T_p^t.

Topological equivalence follows immediately from (8) and the definition. Q.E.D.

The main point of this proposition is that we can essentially simplify the analysis of maps with periodic perturbations. Instead of the original nonautonomous map (4) it suffices to consider one of the autonomous maps $T_1, T_2, \ldots, T_\tau$ defined by (6), (7). The entire dynamics of the original map (4) is thus defined by the set of maps (7), which act independently of each other and are only related by the initial conditions.

Constructions (5)–(7) directly lead to another interesting result.

Lemma 2 [18, 27]. *The period* t *of every cycle of the perturbed map (4) is a multiple of the perturbation period* τ: $t = \tau k$, *where* k *is a positive integer.*

Note that we did not impose any conditions on the set A during the construction of the maps $T_1, T_2, \ldots, T_\tau$ or in our proof. All the results therefore remain valid for any set A of admissible values of the parameter a of the dynamical system (3) with a τ-periodic perturbation.

Introduce the subset $A_c \subset A$ of the set of parameter values such that if $a \in A_c$, then the map (3) has chaotic behavior. It has been shown analytically (see, e.g., [18, 27, 29, 30]) that for $j = 1$ and $j = 2$ periodic perturbations may suppress chaos and stabilize certain cycles of these maps. In other words, it has been shown that for some one-dimensional and two-dimensional chaotic maps there exist perturbations $\widehat{a} = (a_1, a_2, \ldots, a_\tau)$ such that for some $\widehat{a} \in \mathbf{A}_c$ (or $g(a) \in A_c$; see (4)) the perturbed map (4) is regular with a stable cycle of period $t = \tau k$. This result has been proved for a wide class of maps [27, 29, 30]. The emergence of periodic dynamics as a consequence of external perturbations under fairly general conditions on the form of the family of maps is apparently a typical phenomenon.

Below we consider only one-dimensional maps ($j = 1$). For such maps we can generalize the theory developed in [27, 29, 30] and efficiently apply the method to find perturbations that stabilized prespecified cycles in applications. The implementation of this method relies on the following formal result.

Theorem 1. *Assume that the maps* $T_a : x \mapsto f(x, a),\ x \in M,\ a \in A,$ *satisfy the following properties:*

(i) there exists a subset $\sigma \subset M$ *such that for all* $x_1, x_2 \in \sigma$ *there exists a value* $a^* \in A$ *for which* $f(x_1, a^*) = x_2$;

(ii) there exists a critical point $x_c \in \sigma$ *such that*

$$\left.\frac{\partial f(x, a)}{\partial x}\right|_{x=x_c} \equiv \mathrm{D}_x f(x_c, a) = 0$$

for every $a \in A$.

Then for every $x_2, x_3, \ldots, x_\tau \in \sigma$ *there exist* x_1 *and* $a_1, a_2, \ldots, a_\tau$ *such that the cycle* $(x_1, x_2, \ldots, x_\tau)$ *is a stable cycle of the perturbed map* $\mathbf{T}_a$ *for* $\widehat{a} = (a_1, a_2, \ldots, a_\tau)$.

Proof. Take arbitrary $x_1,\ x_2,\ \ldots,\ x_\tau$. By condition (i), the system of equations for the parameters $a_1, a_2, \ldots, a_\tau$

$$\begin{aligned} f(x_1, a_1) &= x_2, \\ f(x_2, a_2) &= x_3, \\ &\ldots\ldots\ldots\ldots\ldots \\ f(x_\tau, a_\tau) &= x_1 \end{aligned} \tag{9}$$

has a solution of the form $\widehat{a} = (a_1, a_2, \ldots, a_\tau)$. This means that the sequence of values $(x_1, x_2, \ldots, x_\tau) = p$ is a cycle of period τ of the map $\mathbf{T}_a$ in the presence of the periodic perturbation $\widehat{a} = (a_1, a_2, \ldots, a_\tau)$. To stabilize this cycle p, it suffices to make the element x_1 close to the critical value x_c, because the multiplier $\beta(p) = \prod_{i=1}^{\tau} \mathrm{D}_x f(x_i, a_i)$ and $\mathrm{D}_x f(x_c, a) = 0$ for all a. This guarantees the stability condition $|\beta(p)| < 1$. Q.E.D.

Families of polymodal maps obviously satisfy conditions (i), (ii) . Since every cycle of the form $(x_c,\ x_2,\ x_3,\ \ldots,\ x_\tau)$ is stable for arbitrary $x_i \in \sigma$, our assertion makes it possible to apply the proposed method for controlling the dynamics of systems that are effectively described by such families.

In a real system, the parameters experience small perturbations from the external environment. Let us examine the robustness of the proposed method under such perturbations. To this end, we will estimate the admissible distortions of the parameter values $(a_1, a_2, \ldots, a_\tau)$ and the cycle elements $(x_1, x_2, \ldots, x_\tau)$. Assume that a perturbation $(a_1,\ a_2,\ \ldots,\ a_\tau)$ corresponds to the stable cycle $(x_c,\ x_2,\ x_3,\ \ldots,\ x_\tau)$. Now suppose that the values a_i change slightly,

$$(a_1', a_2', \ldots, a_\tau') = (a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_\tau + \Delta a_\tau),$$

where $|\Delta a_i| \le \delta_a$. Let us find the maximum admissible δ_a when the perturbed cycle remains stable and investigate how the cycle is distorted, i.e., find Δx_i for

$$(x_1', x_2', \ldots, x_\tau') = (x_c + \Delta x_1, x_2 + \Delta x_2, \ldots, x_\tau + \Delta x_\tau).$$

The results of these computations lead to the following exact bound.

Theorem 2. *Let* $f(x, a) \in C^2[M \times A]$ *and the perturbed map* $\mathbf{T}_a$ *for* $\widehat{a} = (a_1, a_2, \ldots, a_\tau)$ *has a stable cycle of period* τ, $p = (x_1, x_2, \ldots, x_\tau)$. *Under these assumptions, if*

$$|\Delta a_i| \leq \delta_a = \frac{1}{\tau S_a L S_x^{\tau-1} \sum\limits_{i=1}^{\tau} S_x^i},$$

where $i = 1, 2, \ldots, \tau$, $S_a = \max\limits_{x,a} |\mathrm{D}_a f(x, a)|$, $L = \max\limits_{x,a} |\mathrm{D}_x^2 f(x, a)|$, $S_x = \max\limits_{x,a} |\mathrm{D}_x f(x, a)|$, *then this map also has a stable cycle* $p' = (x_c + \Delta x_1, x_2 + \Delta x_2, \ldots, x_\tau + \Delta x_\tau)$ *of period* τ *for* $\widehat{a}' = (a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_\tau + \Delta a_\tau)$, *and* $|\Delta x_i| \leq \delta_x = 1/L S_x^{\tau-1}$.

Proof. Assume that all a_i are perturbed, $a_i' = a_i + \Delta a_i$. Find the increment $\Delta x_1 = x_1' - x_c$. Here x_1' should be a fixed point of the map T_1 (see (7)), i.e., $x_1' = F_1(x_1', a_1', a_2', \ldots, a_\tau')$. Then

$$x_c + \Delta x_1 = F_1(x_c, a_1, a_2, \ldots, a_\tau) + \mathrm{D}_x F_1(x_c, \widehat{a})\Delta x_1 + \sum_{i=1}^{\tau} \mathrm{D}_{a_i} F_1(x_c, \widehat{a})\Delta a_i.$$

Hence, using the relationships $x_c = F_1(x_c, \widehat{a})$ and $\mathrm{D}_x F_1(x_c, \widehat{a}) = \beta(p) = 0$, we find that

$$\Delta x_1 = \sum_{i=1}^{\tau} \prod_{l=i+1}^{\tau} \mathrm{D}_x f(x_l, a_l)\mathrm{D}_a f(x_i, a_i)\Delta a_i.$$

Thus,

$$|\Delta x_1| \leq \delta_a \sum_{i=1}^{\tau} \prod_{l=i+1}^{\tau} \big|\mathrm{D}_x f(x_l, a_l)\big|\big|\mathrm{D}_a f(x_i, a_i)\big| \leq \delta_a \tau S_a \sum_{i=1}^{\tau} S_x^i. \tag{10}$$

Let us estimate the resulting change in the cycle multiplier:

$$\begin{aligned}\beta(p') - \beta(p) = \beta(p') &= \prod_{i=1}^{\tau} \mathrm{D}_x f(x_i', a_i') \\ &= \sum_{i=1}^{\tau} \mathrm{D}_x^2 f(x_i, a_i) \prod_{l=1,\, l\neq i}^{\tau} \mathrm{D}_x f(x_l, a_l)\Delta x_i \\ &\quad + \sum_{i=1}^{\tau} \mathrm{D}_{ax}^2 f(x_i, a_i) \prod_{l=1,\, l\neq i}^{\tau} \mathrm{D}_x f(x_l, a_l)\Delta a_i.\end{aligned}$$

In both sums, only the first terms do not vanish, because $\mathrm{D}_x f(x_1, a_1) = \mathrm{D}_x f(x_c, a_1) = 0$. Therefore,

$$\beta(p') = \big[\mathrm{D}_x^2 f(x_c, a_1)\Delta x_1 + \mathrm{D}_{ax}^2 f(x_c, a_1)\Delta a_1\big] \prod_{l=2}^{\tau} \mathrm{D}_x f(x_l, a_l).$$

However, $\mathrm{D}^2_{ax} f(x_c,\, a_1) = \mathrm{D}_a\left(\mathrm{D}_x\, f(x_c,\, a)\right)\big|_{a=a_1} = \mathrm{D}_a\,(0) = 0$. We thus obtain

$$|\beta(p')| = |\Delta x_1|\left|\mathrm{D}^2_x\, f(x_c,\, a_1)\right| \prod_{l=2}^{\tau}\left|\mathrm{D}_x f(x_l,\, a_l)\right|.$$

Stability of the cycle necessitates the inequality

$$|\Delta x_1|\left|\mathrm{D}^2_x\, f(x_c,\, a_1)\right| \prod_{l=2}^{\tau}\left|\mathrm{D}_x\, f(x_l,\, a_l)\right| \le |\Delta x_1| L S_x^{\tau-1} < 1.$$

Hence we obtain $|\Delta x_1| \le \delta_x = 1/(L S_x^{\tau-1})$.

Thus, if the perturbation Δx_1 is less than δ_x, then the cycle remains stable. But the maximum admissible change Δx_1 when the parameters are perturbed by δ_a is given by inequality (10). Therefore the condition on δ_a may be written as $\delta_a \tau S_a \sum_{i=1}^{\tau} S_x^i = 1/(L S_x^{\tau-1})$ or

$$\delta_a = \frac{1}{\tau S_a L S_x^{\tau-1} \sum\limits_{i=1}^{\tau} S_x^i}.$$

Q.E.D.

3. The Family of Quadratic Maps

Let us now consider the familiar family of quadratic maps. A particular case is the so-called *logistic* map, i.e., the map T_a of the interval $[0,\, 1]$ into itself:

$$T_a\colon\ x \longmapsto \varphi(a,\, x) = ax(1-x). \tag{11}$$

The family (11) models various physical phenomena (see, e.g., [31–33]). It is well known that for $a \in (0,\, a_\infty)$, $a_\infty = 3.569\ldots$, this map has regular dynamics: a stable cycle of period $t = 2^k$. However, for $a \in (a_\infty,\, 4]$, the map T_a may display both regular and chaotic behavior. It is known [34, 35] that the set A_c corresponding to chaotic behavior of the map (11) is of positive Lebesgue measure and the point $a = 4$ is a density point of this set.

Consider the parametrically perturbed map (11). If the perturbation period is τ, then it can be written in the form

$$x_{n+1} = a_n x_n (1 - x_n), \qquad a_n = a_{n \bmod \tau + 1}. \tag{12}$$

It has been previously shown [18, 27, 30] that the perturbations $\widehat{a} = (a_1,\, a_2,\, \ldots,\, a_\tau)$ acting only on the chaotic set $\mathbf{A}_c$ may stabilize its dynamics. In other words, we have the following exact result.

Theorem 3 [18]. *There exists a subset $\mathbf{A}_d \subset \mathbf{A}_c$ of the set of all perturbations acting on A_c such that if $\widehat{a} \in \mathbf{A}_d$, then the perturbed map (12) has a stable cycle.*

The set of all perturbations $\mathbf{A}_d$ leading to regular dynamics has been investigated in more detail in [5]. It has been shown that the parameter values corresponding to stable cycles have a neighborhood at least of order $\sim 10^{-5}a$. Moreover, it has been established by numerical analysis that there are no cycles of period $t = 2$ for the period-2 perturbation $\tau = 2$ in the region $[3.8,\, 4.0]$. This is a particular case of the results presented below.

The family of maps (11) clearly satisfied the conditions of Theorem 1. For the given map the set σ is the interval $[x_b, x_e]$, where x_b and x_e are the solutions of the equation $x_{\text{int}} = f(x, 4)$, $x_{\text{int}} \neq 0$ is the intersection point of the curve $y = 4x(1-x)$ and the straight line $y = x$. Thus, $[x_b, x_e] = [1/4, 3/4]$.

Now consider a more general case, without making the assumptions of Theorem 1. In other words, we will find the perturbations $\widehat{a} = (a_1, a_2, \ldots, a_\tau)$ when the map (12) has a stable cycle of some period, which is a multiple of the perturbation period τ (see Lemma 2). We first assume that the perturbed map has a cycle of period equal to the perturbation period $t = \tau$, i.e., $p = (x_1, x_2, \ldots, x_t)$. Then the points forming this cycle obey the following system of equations:

$$\begin{aligned} &x_2 = a_1 x_1 (1 - x_1), \\ &x_3 = a_2 x_2 (1 - x_2), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &x_1 = a_t x_t (1 - x_t). \end{aligned} \tag{13}$$

To solve the inverse problem, i.e., find the parameter values when the map (12) has a given cycle $p = (x_1, x_2, \ldots, x_t)$, we have to express the values a_i from system (13):

$$a_1 = \frac{x_2}{x_1(1-x_1)}, \quad a_2 = \frac{x_3}{x_2(1-x_2)}, \quad \ldots, \quad a_t = \frac{x_1}{x_t(1-x_t)}. \tag{14}$$

These a_i are not necessarily contained in the interval $(0, 4]$ for all possible $x_i \in (0, 1)$. However, if this is so, then for every cycle $p = (x_1, x_2, \ldots, x_t)$ we can find parameter values $(a_1, a_2, \ldots, a_t)$ that correspond to the existence of such a cycle in the map (12). In this case, if the multiplier $|\beta(p)| = \left|\prod_{i=1}^{t} a_i(1-2x_i)\right| < 1$, then the cycle is stable. Using Eq. (14), we thus obtain the condition

$$\beta(p) = \left|\prod_{i=1}^{t} \frac{x_{i+1}}{x_i(1-x_i)}(1-2x_i)\right| = \left|\prod_{i=1}^{t} \frac{1-2x_i}{1-x_i}\right| < 1. \tag{15}$$

If the critical point $x_c = 1/2$ is one of the cycle points, then $(1-2x_c)/(1-x_c) = 0$. In this case, inequality (15) always holds.

The set of values $p = (x_1, x_2, \ldots, x_t)$ for which $a_i \in (0, 4]$ and inequality (15) holds forms a certain region in the coordinate space $\mathbb{R}^t$. Each point in this region corresponds to a stable cycle of the perturbed map. Using the system of equations (14), we can construct the corresponding region in the parameter space $\mathbb{R}^t$.

As an example, consider a perturbation of period $\tau = 2$. By Lemma 2, the cycle of the perturbed map (12) may only have the period $t = 2k$ for some integer $k \geq 1$. Let us investigate the existence domain of such stable cycles in the coordinate and parameter spaces for $k = 1, 2, 3$.

I. $k = 1$. Then the perturbation period $\tau = 2$ is equal to the stable-cycle period $t = \tau = 2$. It is easy to see that in the space (x_1, x_2) its existence domain is defined by the following system of inequalities:

$$0 < \frac{x_2}{x_1(1-x_1)} \leq 4, \qquad 0 < \frac{x_1}{x_2(1-x_2)} \leq 4, \qquad \left|\frac{1-2x_1}{1-x_1}\frac{1-2x_2}{1-x_2}\right| < 1. \tag{16}$$

The solution of the first and second inequalities corresponds to the set of all admissible values of period 2. The third inequality identifies in this set the existence domain of stable cycles. Take $x_1 \in (0, 1)$. Then solving system (16)

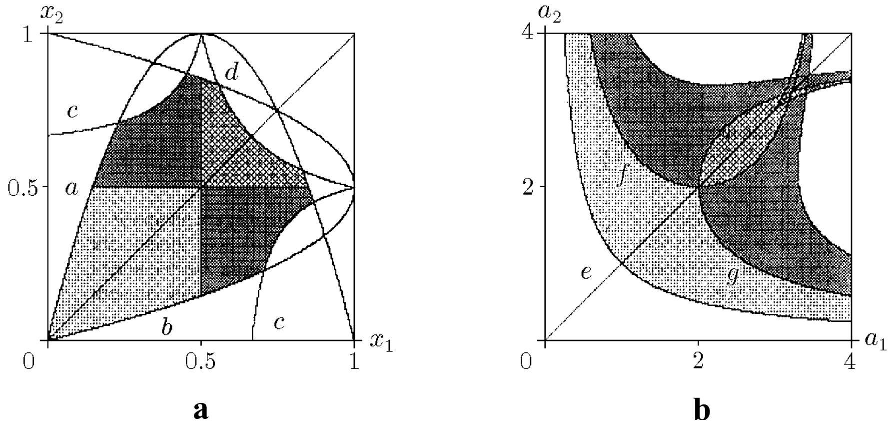

Fig. 1. Existence domain of stable cycles of period 2 for a perturbed ($\tau = 2$) quadratic map defined by the curves $x_2 = 4x_1(1-x_1)$ (a), $x_1 = 4x_2(1-x_2)$ (b), $x_2 = (3x_1-2)/(5x_1-3)$ (c), $x_2 = x_1/(3x_1-1)$ (d) in the space (x_1, x_2) and the curves $a_2 = 1/a_1$ (e), $a_2 = 8/[a_1(4-a_1)]$ (f), $a_1 = 8/[a_2(4-a_2)]$ (g) in the parameter space (a_1, a_2).

for x_2, we obtain

$$0 < x_2 < \frac{3x_1-2}{5x_1-3}, \qquad 0 < x_1 < \frac{1}{3}, \qquad 0 < x_2 < \frac{x_1}{3x_1-1}, \qquad \frac{1}{3} < x_1 < \frac{3}{5},$$

$$\frac{3x_1-2}{5x_1-3} < x_2 < \frac{x_1}{3x_1-1}, \qquad \frac{3}{5} < x_1 < 1.$$

This result is presented in Fig. 1a.

To construct the corresponding region in the parameter space (a_1, a_2), we need to transform the region in Fig. 1a by relationships (14). Performing this operation, we partition the region in Fig. 1a into subregions, which are mapped one-to-one onto the plane (a_1, a_2) (they are marked by different hatchings). It now remains to transform the boundaries of these subregions. We thus obtain the existence domain of stable cycles of period 2, $p = (x_1, x_2)$, in the parameter space (a_1, a_2) (Fig. 1b).

It is now easy to analyze the perturbed quadratic map.

(i) Since the region in Fig. 1b has intersecting subregions, the map defined by system of equations (13) is single-valued but not one-to-one.

(ii) The presence of intersection subregions indicates that the perturbed map (12) is *bistable*: for certain parameter values it may simultaneously have two stable cycles of period 2.

(iii) The parameter region [3.8, 4.0] does not intersect with the region of parameter values corresponding to cycles of period 2 (Fig. 1b). This explains why no cycles of period 2 were found in [18] (see above).

II. $k = 2$. In this case, the stable cycle is of period 4, i.e., $p = (x_1, x_2, x_3, x_4)$, and the perturbation is defined by two parameters (a_1, a_2), as previously. Let us determine the parameter values a_1 and a_2 when these cycles exist and are stable.

From (13),

$$\begin{aligned} x_2 &= a_1 x_1 (1 - x_1), \\ x_3 &= a_2 x_2 (1 - x_2), \\ x_4 &= a_1 x_3 (1 - x_3), \\ x_1 &= a_2 x_4 (1 - x_4). \end{aligned} \tag{17}$$

This gives

$$\begin{aligned} a_1 &= \frac{x_2}{x_1(1 - x_1)} = \frac{x_4}{x_3(1 - x_3)}, \\ a_2 &= \frac{x_3}{x_2(1 - x_2)} = \frac{x_1}{x_4(1 - x_4)}. \end{aligned} \tag{18}$$

It is easy to see that not every set of values $(x_1,\, x_2,\, x_3,\, x_4)$ corresponds to a perturbed-map cycle. Taking two relationships in (18) as independent, we can analytically express the other two. In the same way we find the parameters a_1 and a_2 in terms of two independent values. Take x_1 and x_3 as independent values. Then, setting $q_1 = x_1(1 - x_1),\ q_3 = x_3(1 - x_3)$, we obtain the system of equations

$$\frac{x_2}{x_4} = \frac{q_1}{q_3}, \qquad 1 - x_2 = (1 - x_4)\frac{x_3 q_3}{x_1 q_1}.$$

Hence we easily express x_4 and x_2 in terms of x_1 and x_3:

$$x_4 = \frac{x_1 q_1 q_3 - x_3 q_3^2}{x_1 q_1^2 - x_3 q_3^2}, \qquad x_2 = \frac{x_1 q_1^2 - x_3 q_3 q_1}{x_1 q_1^2 - x_3 q_3^2}.$$

We can also express a_1 and a_2 in terms of x_1 and x_3:

$$a_1 = \frac{x_1 q_1 - x_3 q_3}{x_1 q_1^2 - x_3 q_3^2}, \qquad a_2 = \frac{x_1}{a_1 q_3 (1 - a_1 q_3)}. \tag{19}$$

Relationship (19) may be applied to construct the existence domain of a stable cycle of period 4 in the parameter space $(a_1,\, a_2)$. Indeed, choosing arbitrary $x_1,\ x_3$, we find $a_1,\ a_2$ and also compute $x_2,\ x_4$. In what follows we only take x_1 and x_3 that satisfy the following relationships:

$$\begin{gathered} 0 < a_1 \le 4, \qquad 0 < a_2 \le 4, \\ \beta(p) = \left| \frac{1 - 2x_1}{1 - x_1} \frac{1 - 2x_2}{1 - x_2} \frac{1 - 2x_3}{1 - x_3} \frac{1 - 2x_4}{1 - x_4} \right| < 1. \end{gathered} \tag{20}$$

Figure 2 shows the region defined in the space $(a_1,\, a_2)$ by conditions (20) combined with (19). We see the intersection regions of the separate "branches" that correspond to the bistable behavior of the perturbed map. Note the similarity of these "branches" (see Fig. 2 and Fig. 1b), i.e., there exist scaling transformations that take the region of Fig. 1b into a subregion in Fig. 2. Such scale invariance also holds for $k = 3$.

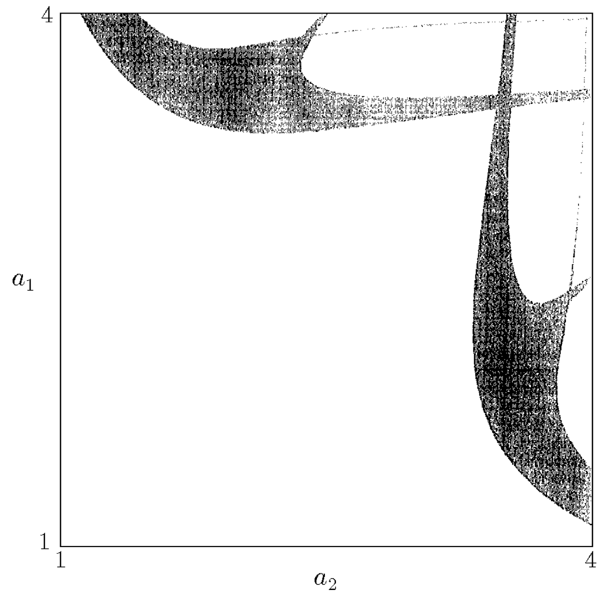

Fig. 2. Existence domain of stable cycles of period 4 for a perturbed ($\tau = 2$) quadratic map.

III. $k = 3$. Here the stable cycle of the perturbed map (12) is of period 6, i.e., $p = (x_1,\, x_2,\, x_3,\, x_4,\, x_5,\, x_6)$. Since the perturbation is defined by two parameters $(a_1,\, a_2)$, as before, the points of the cycle p should satisfy the following relationships:

$$a_1 = \frac{x_2}{x_1(1-x_1)} = \frac{x_4}{x_3(1-x_3)} = \frac{x_6}{x_5(1-x_5)},$$
$$a_2 = \frac{x_3}{x_2(1-x_2)} = \frac{x_5}{x_4(1-x_4)} = \frac{x_1}{x_6(1-x_6)}. \tag{21}$$

There are four expressions linking the values of the coordinates $(x_1,\, x_2,\, x_3,\, x_4,\, x_5,\, x_6)$ of the cycle p. Taking two as independent, we can obtain the other two and use them to express the parameters a_1 and a_2. Unlike the case $k = 2$, this procedure cannot be completed analytically.

Let us briefly consider what can be obtained from relationships (21). First, as for $k = 2$, we take x_1 and x_3 as the independent coordinates. Then, applying the same transformations as in the previous case, we obtain

$$a_1 = \frac{x_3 q_3 - x_5 q_1}{x_3 q_3^2 - x_5 q_1^2} = \frac{x_1 q_1 - x_3 q_5}{x_1 q_1^2 - x_3 q_5^2}, \tag{22}$$

where, as before, $q_i = x_i(1-x_i)$. Equation (22) is simply a relationship among $x_1,\ x_3,\ x_5$. Using (21), we obtain

$$A x_5^5 - B x_5^4 + C x_5^3 - D x_5^2 + E x_5 - F = 0, \tag{23}$$

where

$$A = q_1, \qquad B = 2q_1 + x_3 q_3, \qquad C = q_1 + q_1^2 + 2x_3 q_3,$$

$$D = x_3 q_3^2 + x_3 q_3 + q_1^2, \qquad E = x_3 q_3^2, \qquad F = x_1 q_1 q_3 (q_3 - q_1).$$

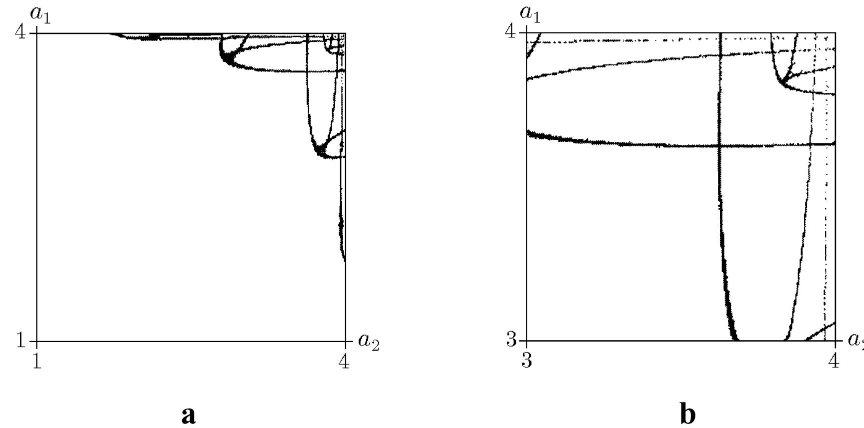

Fig. 3. Existence domain of stable cycles of period 6 for a perturbed ($\tau = 2$) quadratic map: for the parameter interval $[1, 4]$ (a) and a more detailed picture for the region $[3, 4]$ (b).

Now find $x_5 = f(x_1, x_3)$ from Eq. (23) and use (21) and (22) to obtain all the remaining cycle parameters: a_1, a_2, x_2, x_4, x_6. Choosing among all the resulting cycles only those for which $x_i \in (0, 1)$, $i = 1, 2, \ldots, 6$, a_1, $a_2 \in (0, 4]$, and $|\beta(p)| < 1$, we construct the existence domain of a stable cycle of period 6 for the perturbed map (12) in the parameter space (a_1, a_2). These results are presented in Fig. 3. We see from the figure that the subregions have the same typical structure as in Fig. 1b. Moreover, the existence domains of stable cycles of periods 4 and 6 in the presence of perturbations of period 2 intersect with the chaotic behavior region [3.8, 4.0]. This property substantiates the numerical results of [5], where stable cycles of periods 4 and 6 have been observed in the presence of perturbations of period 2.

Let us now proceed with a numerical analysis of the perturbed map (12) and construct the bifurcation diagram in the parameter space (a_1, a_2). The general form of such a diagram is shown in Fig. 4. We clearly distinguish the regions of periods $t = \tau$, 2τ, 3τ that have been obtained analytically in Figs. 1–3.

Figure 5 presents a more detailed part of the bifurcation diagrams in the parameter region $[3.5, 4]$. The previously noted scale invariance is clearly seen. The simply connected regions corresponding to stable cycles of specified periods have the typical "swallow tail" shape, with self-intersecting "tails". A detailed analysis of the entire diagram shows that a similar picture is observed when the scale is increased further.

4. The Family of Circle Maps

Let us now consider the standard sine map of a circle, which effectively describes the transition from quasiperiodic motion to chaos in nonlinear systems [36, 37]:

$$T_{a,b}\colon\ x \mapsto \varphi\,(a,\, b,\, x) = a + x + b \sin x \quad (\mathrm{mod}\, 2\pi), \tag{24}$$

where a and b are the control parameters. Circle maps arise in many problems in physics, chemistry, and biology; for instance, the periodically excited Josephson junction [38], some problems of chemical kinetics [39], unordered contractions of the cardiac muscle [6–9], and others [40, 41].

For $b < 1$ the function φ is monotone increasing on $(0; 2\pi)$ (and therefore one-to-one). The dynamics of the map (24) is therefore determined by the rotation number, which may be defined as

$$\rho = \lim_{n\to\infty} \frac{1}{2\pi} \frac{\varphi^{(n)}\,(x_0) - x_0}{n},$$

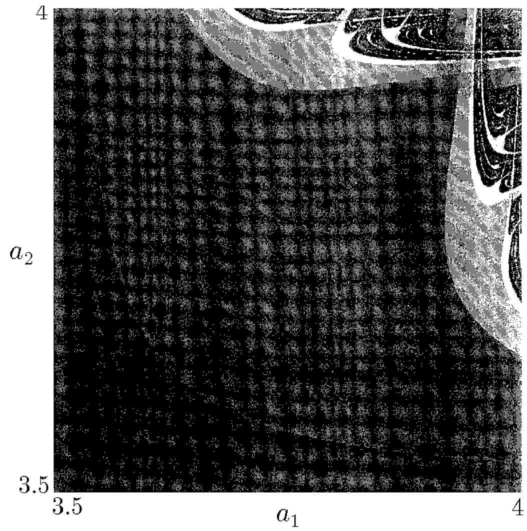

Fig. 4. Bifurcation diagram of map (12).

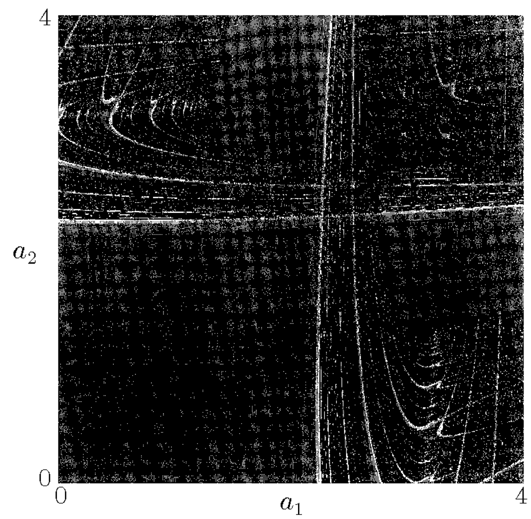

Fig. 5. Bifurcation diagram of map (12) for parameter values in $[3.5,\, 4]$.

where $\varphi^{(n)}(x) = \underbrace{\varphi \circ \varphi \circ \ldots \circ \varphi(x)}_{n}$ is the nth iteration of the map. Depending on the rotation number ρ — whether rational or irrational — the circle map (24) displays periodic or quasiperiodic dynamics, respectively. For $b > 1$, the function φ is no longer a diffeomorphism, and the map (24) is not one-to-one. In this case, given certain relationships between the parameters a and b, the map (24) does not have stable periodic trajectories. Instead, it displays chaotic behavior and is characterized by a positive Lyapunov exponent [42, 43].

Take $b = \mathrm{const} > 1$ and choose a as the perturbation parameter. Then the perturbed map may be written in the form

$$\begin{aligned} x_{n+1} &= a_n + x_n + b\sin x_n, \quad (\mathrm{mod}\, 2\pi), \\ a_{n+1} &= a_{n \bmod \tau}, \end{aligned} \tag{25}$$

where τ is the perturbation period. Consider the case $\tau = 2$. Let us find the conditions when the perturbed map (25) has a stable cycle whose period is equal to the perturbation period, i.e., $t = \tau = 2$. We will investigate the existence regions of these stable cycles in both the coordinate space and the parameter space. As for a quadratic map, we solve the inverse problem. Assume that the points x_1 and x_2 form a cycle of period 2, $p = (x_1, x_2)$. Then they obey the following system of equations:

$$\begin{aligned} x_2 &= a_1 + x_1 + b \sin x_1 \pm 2\pi k, \\ x_1 &= a_2 + x_2 + b \sin x_2 \pm 2\pi m, \end{aligned} \tag{26}$$

where k, m are integers, $k = 0, 1, 2, \ldots$, $m = 0, 1, 2, \ldots$. These integers are introduced to allow for the fact that the image point may complete several turns around the circle during one iteration. System (26) is used as a condition to find the parameters a_1 and a_2. We also need to write the stability condition for the cycle $p = (x_1, x_2)$:

$$|(1 + b \cos x_1)(1 + b \cos x_2)| < 1. \tag{27}$$

Note that equality (27) is independent of the parameters a_1 and a_2, and also of the values k and m. It is therefore sufficient to construct the existence region of stable cycles $p = (x_1, x_2)$ in the coordinate space (x_1, x_2) using condition (27), and then apply Eqs. (26) to transform it into the corresponding region in the parameter space (a_1, a_2) for various k and m.

We will consider the simplest case, when $k = m = 0$. To justify this choice, it suffices to note that a region constructed in the parameter space yields all other regions for other k and m by $\pm 2\pi N$ horizontal and vertical translations ($N = 0, 1, 2, \ldots$).

Given these remarks, we rewrite system (26) in the form

$$\begin{aligned} x_2 &= a_1 + x_1 + b \sin x_1, \\ x_1 &= a_2 + x_2 + b \sin x_2, \end{aligned} \tag{28}$$

If condition (27) is satisfied for the points (x_1, x_2), then the cycle $p = (x_1, x_2)$ is stable. Inequality (27) splits into the following set of inequalities:

$$\begin{aligned} &\cos x_2 < -\frac{b \cos x_1}{b^2 \cos x_1 + b}, \qquad \cos x_2 > -\frac{2 + b \cos x_1}{b^2 \cos x_1 + b}, \qquad b^2 \cos x_1 + b > 0, \\ &\cos x_2 > -\frac{b \cos x_1}{b^2 \cos x_1 + b}, \qquad \cos x_2 < -\frac{2 + b \cos x_1}{b^2 \cos x_1 + b}, \qquad b^2 \cos x_1 + b < 0. \end{aligned} \tag{29}$$

To obtain an explicit dependence of x_2 on x_1, we need to take arccos of the left- and right-hand sides of each inequality, which have the form cos $\cos x_2 < F(b, x_1)$ or $\cos x_2 > F(b, x_1)$, where $F(b, x_1)$ is a function. We should bear in mind that $\arccos$ of the right-hand side exists if

$$|F(b, x_1)| < 1.$$

Further analytical investigation of system (29) has shown the existence of two critical values of the parameter b:

$$b_{c1} = \sqrt{2}, \qquad b_{c2} = 2.$$

The shape of the region is qualitatively different on two sides of these critical values.

Let us consider in more detail the case when $b \in (1,\, b_{c1}) = (1,\, \sqrt{2})$. Solving system (29) for x_2 as a function of x_1 and noting that $x_1,\, x_2 \in (0;\, 2\pi)$, we obtain

$$\begin{gathered} F\,(b,\, x_1) < x_2 < 2\pi - F\,(b,\, x_1) \quad \text{for} \quad 0 < x_1 \leq A \ \text{ and } \ 2\pi - A \leq x_1 < 2\pi, \\ 0 < x_2 < 2\pi \quad \text{for} \quad A < x_1 < 2\pi - A, \end{gathered} \tag{30}$$

where

$$A \;=\; \arccos\left(-\frac{1}{b+1}\right),$$

$$F\,(b,\, x_1) \;=\; \arccos\left(-\frac{b\cos x_1}{b^2\cos x_1 + b}\right).$$

System (30) defines the existence region of stable cycles of period 2, $p = (x_1,\, x_2)$, for the perturbed map (25) with $b \in (1;\, b_{c1})$.

This result is shown in Fig. 6a. For $b_{c1} \leq b < b_{c2}$ and $b \geq b_{c2}$, the expressions describing the relevant region are fairly complex and are therefore omitted. Figures 6b and 6c show the region constructed in the coordinate space $(x_1,\, x_2)$ for various values of the parameter b.

To solve the inverse problem, i.e., find the perturbations of the parameter $\widehat{a} = (a_1,\, a_2)$ for which the map (25) has the specified cycle $p = (x_1,\, x_2)$, we need to express a_i from system (26) as

$$\begin{gathered} a_1 = x_2 - x_1 - b\sin x_1, \\ a_2 = x_1 - x_2 - b\sin x_2. \end{gathered} \tag{31}$$

Under relationships (31) the regions in Figs. 6a–6c transform into corresponding regions in the parameter space $(a_1,\, a_2)$. As in the previous example (see Sec. 3, I), the construction of a one-to-one map from the plane $(x_1,\, x_2)$ onto the plane $(a_1,\, a_2)$ in each case requires partitioning in a certain way the original coordinate-space region into subregions and transforming their boundaries. We thus obtain in the parameter space $(a_1,\, a_2)$ the existence region of stable orbits of period 2, whose structure and partition into subregions are shown in Figs. 6d–6f for various values of the parameter b.

The numerical construction of the general bifurcation pattern of the existence of stable cycles of various periods in the presence of a parametric perturbation of period 2 is shown in Figs. 7a–7c for some values of the parameter b. Figure 7d is the magnified picture of the bifurcation fragment from Fig. 7a. Comparing Figs. 7b and 7d we note that the observed structure of bifurcation regions is characterized by repetition of geometrical features on different scales (see, e.g., [44]). This indicates that the existence regions of stable cycles of different periods are generated and deformed in a certain way. We again have scale invariance, as for the logistic map.

Analysis of our results leads to the following conclusions regarding the family of circle maps:

(i) The presence of intersecting subregions in Figs. 6d–6f indicates that the map defined by the system of equations (31) is an endomorphism, i.e., single-valued, but not one-to-one.

(ii) The perturbed map (25) is multistable.

(iii) There exist scaling transformations that transform certain regions of phase diagrams of the perturbed map (25) in a certain way, i.e., scale invariance is observed.

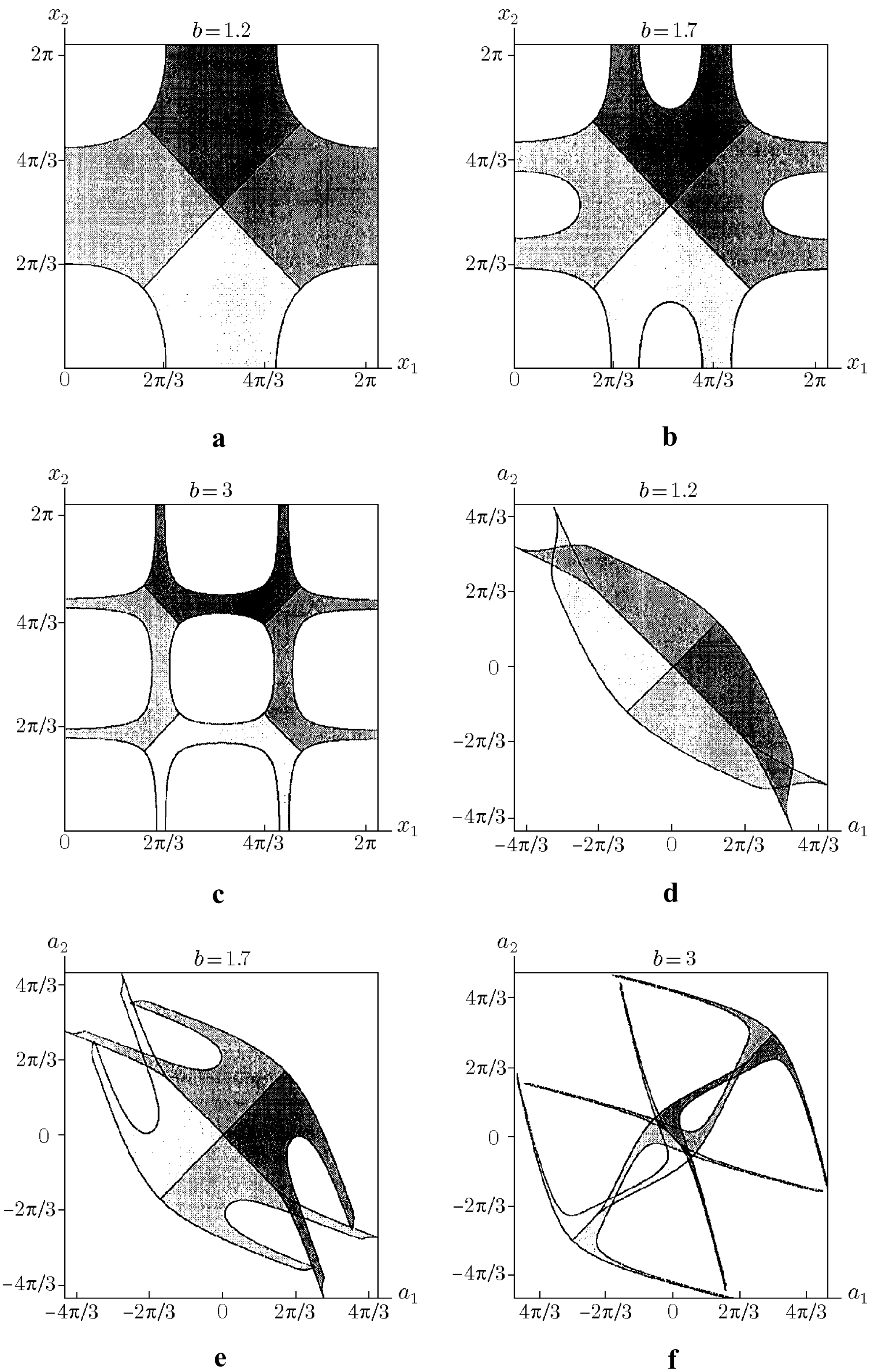

Fig. 6. Existence region of stable cycles of period 2 for perturbed ($\tau = 2$) circle map (25) in the coordinate space (a–c) and in the parameter space (d–f) for various values of the parameter b.

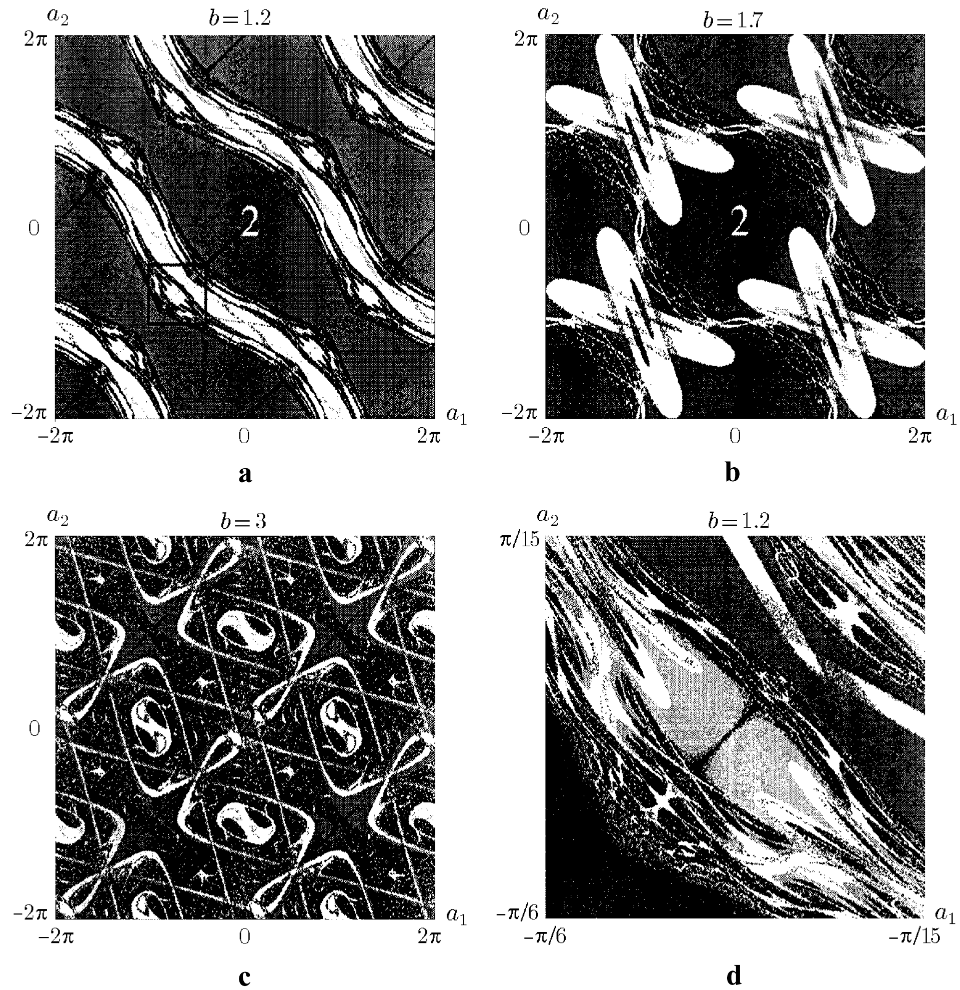

Fig. 7. Bifurcation diagrams of the perturbed circle map (25): (a) $b = 1.2$; (b) $b = 1.7$; (c) $b = 3$; (d) magnified picture of the square outlined in (a); the numeral 2 marks the existence region of cycles of period 2.

5. Concluding Remarks

We have considered the general properties of parametrically perturbed maps. It has been shown that the analysis of such maps can be substantially simplified. Instead of the original nonautonomous map it suffices to consider one of the autonomous maps that are constructed by transposing the functions defining the parametric perturbation. The period of every cycle of the perturbed map is always a multiple of the perturbation period.

For one-dimensional polymodal maps we have derived the conditions when these maps have prescribed dynamics. We have thus obtained a general solution for the control problem for systems that are effectively described by polymodal transformations.

We have studied in detail the dynamics of families of one-dimensional quadratic maps and circle maps in the presence of a periodic perturbation in the parameter. We have shown that for perturbations of period 2 the behavior of the family qualitatively changes. This is manifested in two basic facts.

1. The perturbed map is multistable.

2. For parameter values from the chaotic region of the original map, the perturbed system becomes regular.

This is manifested in the creation of stable cycles of a small period. Moreover, these dynamics-changing processes are stable in the sense that the sets of values of the perturbed parameters form certain regions. These properties suggest that small perturbations (which are generally ignored during modeling) may lead to qualitative changes in the behavior of the system.

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