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Dynamics control of chaotic systems by parametric destochastization

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Abstract. It is shown, that deterministic noise (chaos) appearing via destruction of the quasiperiodic motion may be easily suppressed by weak parametric perturbation of the system.

1. Introduction

It is well known that dynamical chaos is a widespread phenomenon. In this connection a problem of the control and deterministic prediction of the behaviour of real systems with chaotic dynamics on the basis of simulation modelling is very relevant. The problem is that while regular behaviour of the systems is stable with respect to small noise, an exponential instability appears in the chaotic regime, so that any finite perturbations will increase with time. For a model these perturbations can appear when some processes in real systems are not taken into account and parameter values are not strictly determined. Even if all processes are clarified and an absolutely adequate model constructed, it is practically impossible to forecast the behaviour of a system with chaotic dynamics on the basis of such a model. Inevitable mistakes in the initial conditions lead to deviation of an expected trajectory from the actual one after a certain time. Therefore, at this point probabilistic or statistical approaches are needed. However, the probabilistic approach generally does not assure the necessary accuracy of description. The statistical analysis is used when there is a sufficiently large number of identical isolated systems or if a process being investigated may be repeated many times. In the case of complex systems or processes, such an approach seems to be absolutely unacceptable.

Thus, for the control and prediction of chaotic systems behaviour other ways of investigation should be used (see e.g. Farmer and Sidorovich 1987, Castagli 1989, Castagli *et al* 1991, Giona *et al* 1991). In the present article we propose the following approach: if deterministic chaos is a fundamental hindrance and its presence is inevitable then it is necessary to get rid of the chaos. Such a problem is solved within a framework of powerful action: the chaos may be suppressed by an external force (see e.g. Neymark and Landa 1991, Dykman *et al* 1991), for example, of the type $F\cos\omega t$, where F is the amplitude and ω is the frequency of the external force. Systems with an external power action of other forms with the purpose of controlling their dynamics were investigated as well (Jackson and Hubler 1990, Breeden *et al* 1990, Jackson 1991,

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Basti *et al* 1991, Jackson and Kodogeorgiou 1992). However, the power perturbation is often physically unrealizable. Besides, such a solution to the problem of chaos suppression in many cases is inadmissible since it can lead to degeneration of the system to a dimension which is less by one or even more, or to transition of the system into an undesirable state. In contrast to a power action we consider a *weak parametric* influence on the chaotic system at which the chaos is suppressed. This phenomenon was detected numerically for certain class of biosystems (Alekseev and Loskutov 1985a, b) and for the Rossler system (Loskutov 1987) and was called *the phenomenon of parametric destochastization*. Results of parametric periodic influence on a system with chaotic behaviour aiming at the suppression of chaos are also analysed for the Lorenz system (Gribkov and Kuznetsov 1989), for the Duffing–Holmes oscillator (Lima and Pettini 1990) and other models (Hubler 1989, Pettini 1990). Another approach allowing stabilization of unstable periodic orbits involved in a chaotic attractor was proposed in the work by Ott *et al* (1990).

The present paper is organized as follows. The phenomenon of parametric destochastization for flows and for maps is discussed in detail. Then, on the basis of numerical investigations of a circle map, a search algorithm of the necessary conditions at which one can observe parametric destochastization is proposed. Also a possible theoretical explanation of this phenomenon is described.

2. Phenonenon of parametric destochastization

In this part the key idea of parametric chaos suppression is presented. Assume that a system of equations describing a certain process has the form:

$$\dot{y}=V(y,\beta) \qquad y=\{y_1,\ldots,y_n\} \qquad V=\{V_1,\ldots,V_n\} \tag{1}$$

where β is some parameter. Suppose that in the range

$$\beta'\leqslant\beta\leqslant\beta'' \tag{2}$$

equations (1) exhibit a chaotic dynamics caused by a strange attractor. Here and henceforward the term 'strange attractor' means an attractor which is not the finite union of submanifolds of the phase space of the system (Anosov and Arnold 1988). Trajectories on such an attractor are unstable almost everywhere so that any two which are close diverge exponentially. If the initial conditions of the system with the strange attractor are given with some accuracy, then information about its evolution is lost after a time time $t>t_{\text{mix}}$, where t_{mix} is the mixing time (Mikhailov and Loskutov 1991). The idea of parametric destochastization is the following: to find certain parametric perturbations superimposed on system (1) (with conditions (2)), such that it is compelled to exhibit a regular behaviour. In other words one should find some law of variation of parameter β in the region (2) in such a way that the strange attractor transforms into a simple one, for example, into a limit cycle.

As has been shown numerically (Alekseev and Loskutov 1985, Loskutov 1992a), a simple condition which leads to the appearance of the destochastization phenomenon is the following:

$$\beta=\beta_0+\beta_1\sin(\omega t) \tag{3}$$

where ω is a certain frequency, t is time, and

$$\beta_1=(\beta''+\beta')/2 \qquad |\beta_1|\leqslant\beta''-\beta'/2. \tag{4}$$

The last conditions guaranty that the quantity β found from (3) will always remain in the region of chaoticity (2).

Strictly speaking, chaotic dynamics of the system (1) will not exist at every point of the region (2): the vicinity of any point in (2) will always contain values of the parameter β so that they give birth to regular behaviour of the system (1), but only if an attractor is not hyperbolic or stochastic (Afraimovich 1989). A hyperbolic attractor is characterized by the property of structural stability; it is a coarse hyperbolic subset. Systems with the hyperbolic attractor have the most pronounced chaotic properties. Small perturbations of such systems do not lead to qualitative reformations both of the attractor itself and the system dynamics. However, systems with a hyperbolic attractor are hypothetical models of structural stable physical systems with rigorous chaotic properties (Sinai 1989, Afraimovich 1989).

A stochastic attractor is a structural unstable subset of the phase space with densely saddle periodic orbits everywhere. The Lorenz attractor at $b=8/3$, $\sigma=10$, $r=28$ (Bunimovich and Sinai 1979), the Lozi attractor (Lozi 1978), the Belykh attractor (Belykh 1982) and some other attractors belong to the class of stochastic attractors. Sufficiently small perturbations of the system with a stochastic attractor can lead to modification of such an attractor but at the same time the system dynamics remains chaotic. However, with a variation of the parameters of the system, the stochastic attractor can transform into a quasistochastic attractor. These transformations take place for the Lorenz system at $b=8/3$ $\sigma=10.2$, $r=30.2$ (see Afraimovich *et al.* 1980). The overwhelming majority of attractors in chaotic dynamical systems are quasistochastic attractors (Shilnikov 1991). A quasistochastic attractor contains saddle periodic orbits and stable periodic orbits, but with a small attraction basin. Small perturbations of systems with a quasistochastic attractor lead to complex qualitative changes in system dynamics and in the structure of the attractor. That is the reason why, for most systems, the region of chaoticity (2) contains subregions with regular dynamics. In applications however, this circumstance does not play an essential role since the stable orbits contained in the quasistochastic attractor have not been revealed numerically (Afraimovich 1989). The system dynamics with a quasistochastic attractor also looks chaotic.

To clarify the reasons for chaos suppression it is more convenient to analyse maps instead of considering the differential equations (1). For this purpose let us insert in the phase space of the system (1) some $(n-1)$-dimensional hypersurface S (a hyperplane for example) and let us analyse the points of intersection of the phase trajectory with S. Then some set of points $A, B, C, \ldots$ on the surface S will be obtained. Since in S the phase trajectory is connected with these points in a unique way then one can introduce in the surface S some function ϕ that maps (converts) point A into the point B, i.e. $B=\phi(A)$, point B into the point C, i.e. $C=\phi(B)=\phi(\phi(A))$, and so on. Function ϕ is the so-called return map, or Poincare map.

Introduce local coordinates on the surface S. Then the Poincare map is represented as a function that gives the intersection point $\boldsymbol{x}_{k+1}$ for the known intersection point $\boldsymbol{x}_k$

$$\boldsymbol{x}_{k+1}=f(\boldsymbol{x}_k, \mu) \tag{5}$$

where μ is a parameter, f is some $(n-1)$-dimensional function, and $\boldsymbol{x}_k=\{x_k^1, \ldots, x_k^{n-1}\}$ is the totality of local coordinates on the surface S corresponding to intersection points. When dissipation in the system (1) is considerable, the intersection points of the phase trajectory form approximately a one-dimensional curve. Then (5) can be presented as a one-dimensional map

$$x_{n+1}=f(x_n,\mu). \tag{6}$$

Variables k in (5) and n in (6) show the numbers of consecutive intersections and can be considered as (discrete) time. For every initial condition x_0 the map (6) generates an infinite sequence of points

$$\begin{aligned} &x_0 \qquad x_1=f(x_0,\mu) \qquad x_2=f(x_1,\mu)=f(f(x_0,\mu),\mu)\equiv f^{(2)}(x_0,\mu),\ldots \\ &x_m=f(x_{m-1},\mu)=\underbrace{f(f(\ldots f(f}_{m\text{ times}}(x_0,\mu),\mu),\ldots,\mu),\mu)\equiv f^{(m)}(x_0,\mu),\ldots. \end{aligned} \tag{7}$$

This sequence may possess fixed points $\tilde{x}$ and k-cycles. A sequence of points (7) such that $x_n=x_{n+k}$, $x_n\neq x_{n+i}$, for $1<i<k$ and any n, is called a k-cycle or a cycle of period k (or a k-periodic orbit) of the map (6). One-cycle is a fixed point. Fixed points are found from the equation $f(\tilde{x},\mu)=\tilde{x}$. Obviously, the k-periodic orbits, of the map $f(x,\mu)$ are formed by the fixed points of the map $f^{(k)}(x,\mu)$, which are different from the points of the map $f^{(i)}(x,\mu)$, $1\leqslant i<k$.

Dynamics of the map (6) may be both regular and chaotic depending on the values μ and x_0. If the dynamics are chaotic then the map (6) has only unstable fixed points and cycles. In this case the sequence (7) will be completely aperiodic, and will not tend to periodic orbits with increase in the iteration number n. The stability of cycles is determined by the value

$$\lambda=\prod_{i=1}^{k}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{\tilde{x}_i}. \tag{8}$$

If $|\lambda|<1$ then the k-cycle is locally stable. As a criterion of the chaoticity of the map a Lyapunov exponent may also be used (see Mikhailov and Loskutov 1991). It is computed from the expression

$$\Lambda(x_0)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\left|\frac{\mathrm{d}f}{\mathrm{d}x_i}\right|. \tag{9}$$

If $\Lambda>0$, then the map generates deterministic aperiodic sequences (7), i.e. it is chaotic. When $\Lambda<0$, the map possesses a stable k-cycle.

Suppose that in the interval

$$\mu'\leqslant\mu\leqslant\mu'' \tag{10}$$

the map (7) has chaotic dynamics. Then for the parametric destochastization it is necessary, inside this interval, to find a certain variation of the parameter μ such that the map

$$x_{n+1}=f(x_n,\mu_n) \qquad \mu_n\in[\mu',\mu''] \tag{11}$$

generates the cycles of finite period. In contrast to the flows (1) the parameter μ for the map (6) should be varied discretely with time n. In turn, μ_n may vary both periodically and by other means. In the former case which corresponds to periodic perturbation (3) of system (1), the sequence of values of the parameter μ_n consists of identical subsequences of the length i: $\mu_{n+i}=\mu_n$, $\mu_{n+k}\neq\mu_n$, $1<k<i$. For the simplest case, $i=2$, the values μ_n form a parametric two-cycle. In order to show periodicity in the map (11) one should demonstrate that among the set of parameters from the

interval $[\mu', \mu'']$, a certain subset exists such that the map (11) with μ_n from this subset has regular dynamics.

3. The reason for chaos suppression

Chaos in dynamical systems often appears via destruction of quasiperiodic motion. The transition to chaoticity in this case, one can describe effectively by the use of a circle map (Shuster 1988, Mackay and Tresser 1986, Kaneko 1986):

$$x \to f(x, \mu) \qquad \mod 2\pi. \tag{12}$$

Here $f(x, \mu) \in [0, 2\pi]$, $f(0, \mu) = f(2\pi, \mu)$ and $f(x + 2\pi) = f(x, \mu) + 2\pi$. A circle map is quite an important model since it is used to simulate a variety of phenomena, for example, the Josephson junction with periodic perturbations (Bak *et al* 1988), periodically stimulated nonlinear oscillators (Mandel and Kapral 1983) with various biological, chemical (Dolnik *et al* 1984, Glass *et al* 1984), and medical (Courtemanche *et al* 1989, Glass and Mackay 1988, Glass 1991) applications.

As the function $f(x, \mu)$, the so-called 'sine' function of the shape $f(x, a, b) = x + a + b \sin x$ is often chosen (where a and b are parameters). In this case the circle map (12) will take the form

$$x_{n+1} = x_n + a + b \sin x_n, \qquad \mod 2\pi. \tag{13}$$

At $b < 1$ the map (13) is a C^∞-diffeomorphism of the circle (figure 1(a)), and its dynamics is sufficiently well known (Arnold 1965, Guckenheimer and Holmes 1990). In particular for any initial point x_0 there exists the number

$$\rho = \lim_{k \to \infty} \frac{1}{2\pi} \frac{f^{(k)}(x_0)}{k}$$

which is called the rotation number. If the rotation number ρ is rational, i.e. $\rho = p/q$ where p and q are integers then the map (13) has an equal number of stable and unstable q-periodic orbits. In the general case, almost all other orbits will be attracted to the stable ones with an increase in the number n, and the dynamics of the circle map will be periodic. If the rotation number ρ is irrational, i.e. $\rho \neq p/q$, then the orbits will be dense everywhere on the circle so that the behaviour of the map (13) will be quasiperiodic.

When $b > 1$ the map (13) is not one-to-one (figure 1(c)). This signifies that the circle map (13) can generate an aperiodic sequence, i.e. can have chaotic dynamics, depending on the initial value and parameters. Such dynamics in the circle map (13) appear due to well-known scenarios: period doubling, crisis or intermittency (Kaneko 1984). Moreover, coexistence of periodic dynamics and chaos or two different types of chaotic behaviour in the parametric region $b > 1$ is possible.

The phase diagram of the circle map is extremely complex. Let $b > 1$. Following the relation (9) let us numerically determine such regions in the plane of parameters (a, b) where $\Lambda > 0$, i.e. the regions where the dynamics of the map (13) is chaotic. Then the ensuing pattern will be observed (figure 2). Analysing this figure one finds that the chaotic regime and the regular regime in the circle map are closely intertwined. (For detailed investigations see Glass and Mackay 1988 and references cited therein).

Rigorous analytical investigations (Boyland 1986) allow one to find the location of

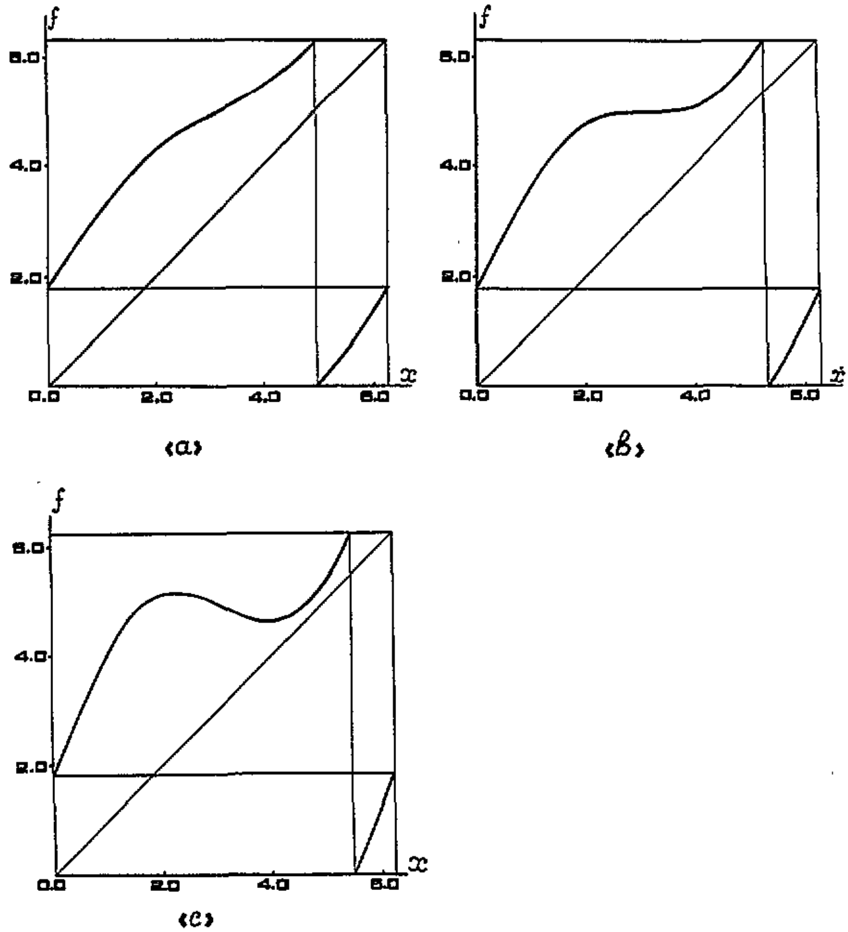

Figure 1. The graph of the circle map (13) at $a=1.8$: the one-to-one case, $b=0.5$ (a); existence of the inflection point, $b=1$ (b); the case of the non-invertible map, $b=1.5$ (c).

an uncountable set of parameter values (a, b), $b>1$, for which the map (13) possesses aperiodic dynamics. Namely, if for the map (13) the values a and b are such that they satisfy a certain functional dependence $a=\Gamma(b)$, and if the lift of the map (13) has a negative Schwarzian derivative, then the circle map (13) has no stable periodic orbits.

Assume now that according to (11) one of the parameters of the map (13) (for example, parameter a) varies with time, i.e. $a=a_n$. This variation may be physically interpreted as a parametric disturbance of the environment. Then for the periodic disturbance with period i the circle map (13) may be rewritten as follows:

$$\begin{aligned}
&x_{in+1}=f_1(x_{in}, a_1, b)\\
&x_{in+2}=f_2(x_{in+1}, a_2, b)\\
&\dots\dots\dots\dots\dots\dots\\
&x_{in+i}=f_i(x_{in+i-1}, a_i, b)\\
&x_{in+i+1}=f_1(x_{in+i}, a_1, b)\\
&\dots\dots\dots\dots\dots\dots \qquad \bmod 2\pi
\end{aligned} \tag{14}$$

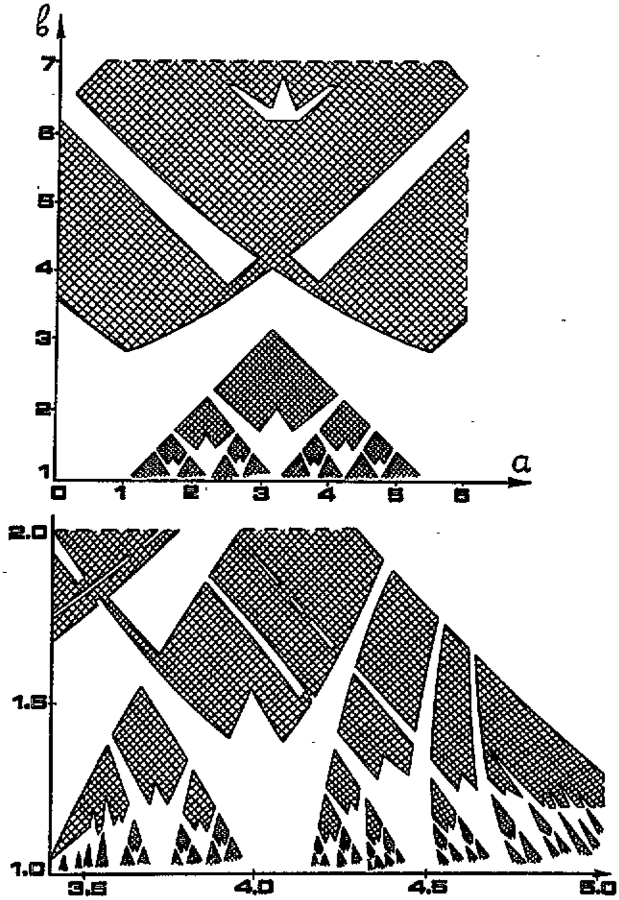

Figure 2. Structure of the parametric space (a, b) of the circle map (13) schematically.

where $f_m(x, a_m, b) = x + a_m + b \sin x$, $m = 1, \ldots, i$. It is easy to see that the sequence of parametric values $a_1, a_2, \ldots, a_i, a_{i+1}, \ldots$ in (14) consists of the totality of subsequences of the length i. According to the condition (11) all these quantities a_1, $a_2, \ldots, a_i$ should correspond to chaotic dynamics of the unperturbed map (13).

Denote the set of values a (b = constant > 1) such that the map (13) has chaotic behaviour by A_c. The set A_c appears when the circle map becomes non-invertible (figure 1b) and has a very complex structure (figure 2). Consequently, in some regions of the parametric space (a, b) small variations of the parameter a, b = constant > 1 (or the parameter $b > 1$, a = constant) lead to a qualitative change in dynamics of the circle map (13). In this sense the circle map (13) is analogous to the system with a quasistochastic attractor. However, in the given case conditions $a_j \in A_c$, $j = 1, \ldots, i$, may be easily satisfied. An aprpropriate method to do this may be based, for example, on the choice of the step Λa for the parameter a (or Λb for the parameter b) in the map (14) so that $a_k - a_{k-1} = \Delta a$ (or $b_k - b_{k-1} = \Delta b$), $a_k \in A_c$, $k = 2, \ldots, i$, have been met. But from iteration to iteration Δa (or Δb) are not necessarily constant.

In the map (14), let $a_j \in A_c$, $j = 1, \ldots, i$. Analysing dynamics in this case it would be expected that the map (14) will exhibit chaotic properties. But this is not true in all cases. The reason is that the cyclic variation from iteration to iteration of the control parameter leads to the appearance of stable cycles. The following assertion explains this.

Let A_c be the set of values of the parameter a, corresponding to chaotic dynamics of the map (13). Let, in (14), $a_j \in A_c$, $j=1, \ldots, i$. Then the set A_c contains certain quantities $a_1^{dn} \in A_c$, $a_2^{dn} \in A_c, \ldots, a_i^{dn} \in A_c$, $n=1, 2, \ldots$ (index d comes from the word 'destochastization') at which the map (14) with $a_1 = a_1^{dn}$, $a_2 = a_2^{dn}, \ldots, a_i = a_i^{dn}$ will generate stable cycles of finite period.

In order to be convinced of the correctness of this assertion it is sufficient to find only one totality of parameters $a_i^{d}, \ldots, a_i^{d}$ such that the map (14) generates stable cycles of finite period and each parameter from this totality corresponds to the chaotic dynamics of the map (13), $a_1^{d}, \ldots, a_i^{d} \in A_c$.

First let us consider the simplest case $i=2$, $b=$constant, $(b>1)$. Then the map (14) may be rewritten as follows

$$\begin{aligned} x_{2n+1} &= f_1(x_{2n}, a_1, b) \\ x_{2n+2} &= f_2(x_{2n+1}, a_2, b) \qquad \bmod 2\pi \end{aligned} \tag{15}$$

where $f_1 = x + a_1 + b\sin x$, $f_2 = x + a_2 + b\sin x$, a_1, $a_2 \in A_c$. Introduce functions of the form $\Phi_1 = f_1(f_2) = x + a_1 + a_2 + b\sin x + b\sin(x + a_2 + b\sin x)$, $\Phi_2 = f_2(f_1) = x + a_1 + a_2 + b\sin x + b\sin(x + a_1 + b\sin x)$. Then it is easy to see that the function Φ_1 corresponds to the odd numbers of the sequence (7) generated by the map (15), and the function Φ_2 corresponds to the even ones. Therefore, determining the initial value x_0 and the quantity $x_1 = f_1(x_0)$, one can rewrite for the map (15)

$$x_{2n+1} = f_1(f_2) = \Phi_1(x_{2n-1}, a_1, a_2, b) \tag{16a}$$

$$x_{2n+2} = f_2(f_1) = \Phi_2(x_{2n}, a_1, a_2, b). \tag{16b}$$

The map (16a) and the map (16b) 'work' independently of each other: their iterations do not connect via x_j, except for the initial values x_0 and x_1 (in contrast to the map (15)).

It is known that for every arbitrary map $g(x)$ any k-cycle is at the same time the fixed points $\bar{x}_j$, $j=1, 2, \ldots, k$, of the map $g^{(k)}(x)$. Generally speaking, the converse statement is not true. This is due to the fact that not only the k-cycles but also the m-cycles, $m=k/i$, $i=2, 3, \ldots, k$, (m is integer) of the map $g(x)$ are the fixed points of $g^{(k)}(x)$. Consequently to ensure that the fixed points $\bar{x}_j$ of the map $g^{(k)}(x)$ form the cycle of period k it is necessary to eliminate these (k/i)-cycles from consideration. This can be arranged by the investigation of $(k-1)$ equations $[g^{(k)}(\bar{x}) - \bar{x}]/[g^{(1)}(\bar{x}) - \bar{x}] = 0$, $1 = 1, 2, \ldots, k-1$. Based upon such reasoning let us consider the functions Φ_1, Φ_2 and the maps (16a, b).

For the reason that the map (15) consists of two consequently performing transformations (16a) and (16b), the $2k$-cycle of the map (15) will give rise to the fixed points $\bar{x}_j^1$ of the map $\Phi_1^{(k)}$ and simultaneously will give rise to the fixed points $\bar{x}_j^2$ of the map $\Phi_2^{(k)}$, $j=1, \ldots, k$. Quite the reverse, if $\Phi_1^{(k)}$ and $\Phi_2^{(k)}$ have the fixed points $\bar{x}_j^1$ and $\bar{x}_j^2$ respectively then it is quite possible (but it is not necessary) that map (15) will have the $2k$-periodic cycle formed by these points. These points will compose such a cycle for the map (15) only if the following equations

$$\begin{aligned} [\Phi_1^{(k)}(\bar{x}^1, a_1, a_2, b) - \bar{x}^1]/[\Phi_1^{(m)}(\bar{x}^1, a_1, a_2, b) - \bar{x}^1] = 0 \\ [\Phi_2^{(k)}(\bar{x}^2, a_1, a_2, b) - \bar{x}^2]/[\Phi_2^{(m)}(\bar{x}^2, a_1, a_2, b) - \bar{x}^2] = 0 \end{aligned} \tag{17}$$

have k common solutions for all $m=1, 2, \ldots, k-1$. Next, the map (15) has regular

Table 1. Some of the stable cycles of the map (14).

i	k	a_i^d	b	$\bar{x}_j^1$	$\bar{x}_j^2$	$\bar{x}_j^3$	$\bar{x}_j^4$	$\bar{x}_j^5$	$\bar{x}_j^6$
2	3	2.832	2.3	4.2610491	0.4430487	—	—	—	—
		2.833		5.5060536	0.8886927				
				5.6673852	5.0244363				
2	4	2.837	2.3	0.6456350	0.3066119	—	—	—	—
		2.849		3.8378215	2.0185224				
				5.4472391	4.8785581				
				6.0296381	5.2117653				
2	4	2.309	2.0	0.1881688	0.6568658	—	—	—	—
		2.313		4.1871416	2.8752882				
				5.0820300	4.7697418				
				5.7106242	5.5301161				
2	5	2.175	1.8	0.7520389	0.7507670	—	—	—	—
		2.176		4.1537265	4.1576714				
				4.8025828	4.8033898				
				5.1858377	5.1858993				
				5.7589478	5.7598357				
6	2	2.281	2.0	6.1074458	1.7565879	6.0051686	1.4570851	5.7371687	0.7204091
		2.282		4.3207936	4.7541909	5.0389381	5.4286282	6.2130677	2.0947620
		2.283							
		2.284							
		2.293							
		2.305							

dynamics when it possesses the stable cycle. Thus, chaos suppression at certain $a_1=a_1^d\in A_c$, $a_2=a_2^d\in A_c$ in the map (15) will be observed not only when map (16a) and (16b) have stable cycles of period k, but also when $\Phi_1^{(k)}$ and $\Phi_2^{(k)}$ have stable fixed points $\bar{x}_j^1$ and $\bar{x}_j^2$, $j=1,\ldots,k$, which are roots of the equations (17). In the latter case, for the chosen number k, the map (15) will, of necessity, possess the stable $2k$-cycle.

However, for the search of the stable cycles of the minimal possible periods there is no need to consider equations (17), since the chaotic map has only the unstable fixed points and the unstable cycles. The appearance of the stable fixed points of the functions $\Phi_1^{(k)}$, $\Phi_2^{(k)}$ at minimal possible number k means immediately that in the map (15) there is a stable cycle of period $2k$ which is formed by these points. Leaning on these arguments we can propose a search algorithm to determine the destochastization parameters $a_1^d\in A_c$, $a_2^d\in A_c$ for the map (15). At the consecutive choice a_1 and a_2 from the chaoticity set A_c all fixed points of the maps $\Phi_1^{(k)}$ and $\Phi_2^{(k)}$ should be considered and among them the stable ones should be determined. If at the given value k these maps do not have stable fixed points then it is necessary to increase the number k by the unit and repeat the described procedure. The investigations are repeated until the values $a_1=a_1^d\in A_c$, $a_2=a_2^d\in A_c$ are found.

The computational techniques for determining the values a_1^d, $a_2^d\in A_c$, however, lead to an analytically intractable problem since the resulting equations for the fixed points are high-order transcendental equations. At the same time these equations do not contain any singularities, and therefore, it is an easy matter to solve them numerically. The numerical method allows one to conclude that there is a certain parameter subset a_1^{dn}, a_2^{dn}, $n=1,2,\ldots,$ such that the maps $\Phi_1^{(k)}(x, a_1^{dn}, a_2^{dn}, b)$ and $\Phi_2^{(k)}(x, a_1^{dn}, a_2^{dn}, b)$ have stable fixed points $\bar{x}_j^1$ and $\bar{x}_j^2$, $j=1,\ldots,k$ (table 1). Consequently the map (15) has a stable cycle of period $2k$ at suitable a_1, a_2 and b.

To show, finally, the correctness of the above assertion it is necessary to show that at the values of the parameters $a(a=a_1^{\mathrm{d}},\ a=a_2^{\mathrm{d}})$ marked on table 1, the map (13) is chaotic. For this purpose it is sufficient to calculate the Lyapunov exponent Λ (see formula (9)). These calculations clearly determine its positive values.

In figure 3 one of the stable cycles of the map (15) is shown. Figure 4 demonstrates the points (set) $a_1^{\mathrm{d}n} \in A_c$, $n=1, 2, \ldots$, corresponding only to the stable cycles of finite periods of the map (15). It is easy to see that the destochastization regions of the circle map (15) are narrow and therefore, the probability of a random hit in them is very small.

Note that at $b>1$, depending on the initial value x_0, the circle map (13) may have two different types of chaos. That is the reason why the regular behaviour of the map (15) at given $a_{1,2}=a_{1,2}^{\mathrm{d}}$ may correspond to not all x_0; it may be both periodic and chaotic. Such a phenomenon takes place, for example, at $a_1^{\mathrm{d}}=2.311$, $a_2^{\mathrm{d}}=2.312$ and $b=2.0$: at $x_0=0.5$ the map (15) has a stable $2k$-periodic cycle ($k=4$), but at $x_0=1.0$ the iterations of (15) do not fall into the attraction basin of this stable cycle, and behaviour of the map is chaotic.

If, in the map (14), $i>2$ then to determine the parameters $a_1^{\mathrm{d}}, \ldots, a_i^{\mathrm{d}} \in A_c$ it is necessary to consider i functions

$$
\begin{aligned}
&\Phi_i = f_i(f_{i-1}(\ldots f_2(f_1)\ldots)) \\
&\Phi_{i-1} = f_1(f_i(f_{i-1}(\ldots f_3(f_2)\ldots))) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\Phi_1 = f_{i-1}(f_{i-2}(\ldots f_1(f_i)\ldots))
\end{aligned}
\tag{18}
$$

where $f_m = x + a_m + b\sin x$, and their kth iterations $\Phi_m^{(k)}$, $m=1, \ldots, i$, and then to find, at certain k, the stable fixed points $\bar{x}_j^1, \bar{x}_j^2, \ldots, \bar{x}_j^i$, $j=1, \ldots, k$. If detection of the parametric destochastization phenomenon is pursued for the arbitrary number k then

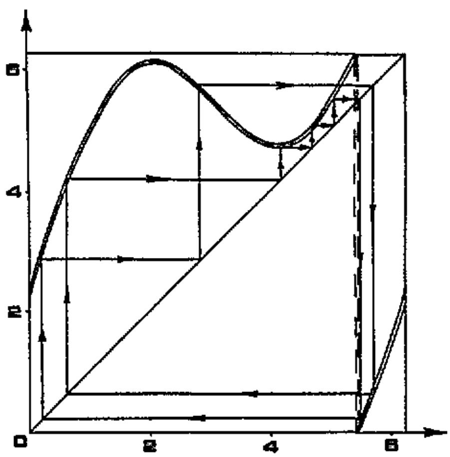

Figure 3. The stable 8-cycle of the map (15) at $b=2.0$, $a_1=2.309$, $a_2=2.313$.

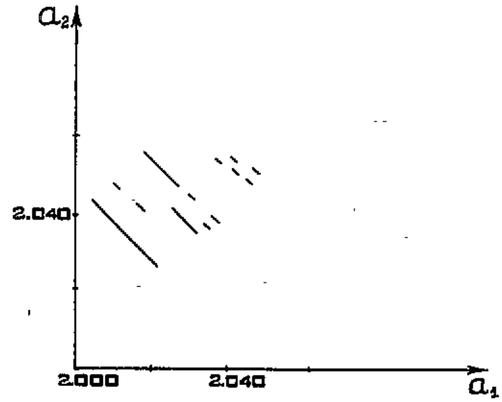

Figure 4. The regions of parametric destochastization for the circle map (15) at $b=2.0$ and $A_c=[2.000, 2.063]$.

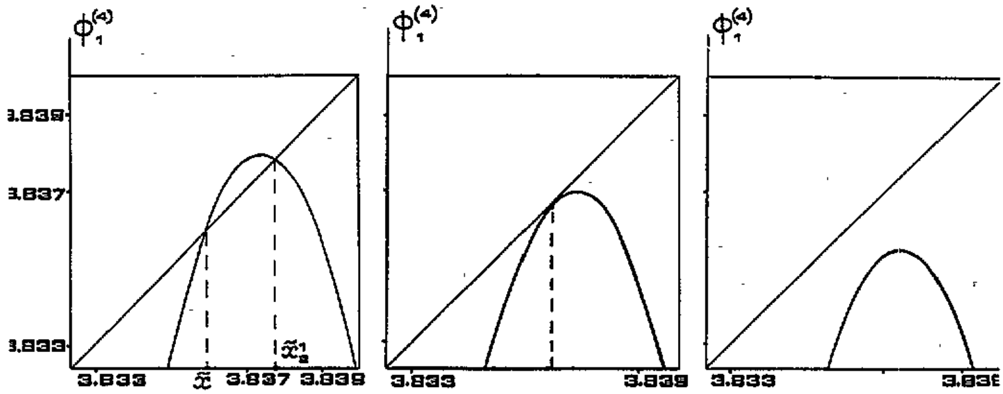

Figure 5. A part of the graph of the function $\Phi_1^{(4)}$ at $b=2.3$ and $a_1=2.837$, $a_2=2.849$ (a), $a_1=2.83698$, $a_2=2.849$ (b), and $a_1=2.83694$, $a_2=2.849$ (c).

in addition to these investigations it is required that the stable fixed points $\bar{x}_j^1, \bar{x}_j^2, \ldots, \bar{x}_j^i$, $j=1, \ldots, k$ satisfy the following equations:

$$
\begin{aligned}
&[\Phi_i^{(k)}(\bar{x}^i, a_1, a_2, \ldots, a_i, b) - \bar{x}^i]/[\Phi_i^{(m)}(\bar{x}^i, a_1, a_2, \ldots, a_i, b) - \bar{x}^i] = 0 \\
&[\Phi_{i-1}^{(k)}(\bar{x}^{i-1}, a_1, a_2, \ldots, a_i, b) - \bar{x}^{i-1}]/[\Phi_{i-1}^{(m)}(\bar{x}^{i-1}, a_1, a_2, \ldots, a_i, b) - \bar{x}^{i-1}] = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \qquad (19) \\
&[\Phi_1^{(k)}(\bar{x}^1, a_1, a_2, \ldots, a_i, b) - \bar{x}^1]/[\Phi_1^{(m)}(\bar{x}^1, a_1, a_2, \ldots, a_i, b) - \bar{x}^1] = 0 \\
&\qquad m = 1, 2, \ldots, k-1.
\end{aligned}
$$

In this case we can be assured that the map (15) will possess a stable $2k$-cycle (table 1).

A question of the loss of stability of the periodic orbits in the map (15) is interesting. If the parameter values a_1, a_2 are varied in such a way that the destochastization phenomenon disappears, then it happens via an inverse tangent bifurcation. Namely, let us consider one of the stable fixed points $\bar{x}_2^1$ which is an element of the stable cycle of the map (15) (figure 5). At variation of a_1 (or a_2) this

Table 2. Some of the stable cycles of the map (14) at $f_m = x + a + b_m \sin x$, $m = 1, \ldots, i$, $b_m > 1$, $b_m \in B_c$.

i	k	a	b_i^d	$\bar{x}_j^1$	$\bar{x}_j^2$	$\bar{x}_j^3$	$\bar{x}_j^4$
2	3	2.8	2.444	1.4811294	0.4440775	—	—
			2.456	4.2940809	4.8498433		
				5.2288950	5.8932804		
2	3	2.8	2.447	0.4429594	1.4818465	—	—
			2.454	4.8516695	4.2917807		
				5.8938495	5.2283660		
3	5	2.8	2.511	0.4371332	0.4257008	0.4344779	—
			2.512	5.0970256	5.0961823	5.0948576	
			2.514	4.2918715	4.3014199	4.2626410	
				5.5643592	5.5667114	5.5678553	
				4.8004426	4.7968967	4.7995006	
4	3	2.8	2.438	1.5013996	0.4513436	1.4712884	0.4438943
			2.439	5.2215370	5.8918994	5.2360619	5.8988027
			2.458	4.4909169	4.8653594	4.3234617	4.8397809
			2.468				

fixed point merges with the unstable point $\tilde{x}$ and then vanishes. A similar situation takes place for other stable points forming the stable cycle under consideration. This explains intermittency, which was described in the works on numerical investigations of destochastization, when the perturbation amplitude β_1 (see (3)) decreases (Alekseev and Loskutov 1987, Loskutov 1987).

The above results lead to the following conclusion.

Period m of the stable cycles generated by the map (14) is not less than i, $m > i$, moreover $m = li$, $l = 1, 2, \ldots$.

It is simple task to show this. For any sequence $x_1, x_2, \ldots, x_n, x_{n+1}, \ldots$ generated by the map (14) every $(in + j - i)$th element, where $j = 1, 2, \ldots, i$, may be considered as the element which is generated by the following map

$$x_{in+j-1} = \Phi_{i-j+1}(x_{in-i+j-1}) \qquad j = 1, 2, \ldots, i \tag{20}$$

where the functions Φ_{i-j+1} are determined by the relations (18). The initial conditions for the (20) are found from the expressions: $x_1 = f_1(x_0)$, $x_2 = f_2(x_1), \ldots, x_{i-1} = f_{i-1}(x_{i-2})$. Every m-cycle of the map (14) is the (m/i)-cycle of the map (20). It is clear that the number (m/i) should be only integer,

Note that if in the map (13) the control parameter b instead of the control parameter a is chosen, i.e. if $b = b(n) \equiv b_n > 1$, $a = \text{constant}$, then the above results (assertion and conclusion) are just the same: at certain values $b_1^d, \ldots, b_i^d \in B_c$ (where B_c is the set of parametric values b, $a = \text{constant}$, at which the map (13) does not have stable periodic orbits) the map (14) with the functions $f_m = x + a + b_m \sin x$, $m = 1, \ldots, i$, will also generate the stable cycles of finite period (table 2).

Thus, we see that the very weak parametric perturbation (sometimes $a_1^d - a_2^d < 10^{-3}$ (!)) may destochastizate a system with complex (chaotic) dynamics; after suppression of the chaos, in principle one can predict its behaviour.

It is important that stability of the points $\bar{x}_k^1, \bar{x}_k^2, \ldots, \bar{x}_k^i$ is not disturbed when small additive random perturbations do not exceed certain threshold action. This is understood by reference to Figure 5(a): the curve $y = \Phi_1^{(k)}$ and the bisectrix $y = x$ cross each other in a typical way. In every particular case the threshold amplitude of the noise action may be estimated from analysis of the intersection depth. For example, for $b=2.3$, $i=2$, $a_1^d=2.832$, $a_2^d=2.833$ this threshold is $\sim 5\times 10^{-6}$.

4. Discussion

Chaotic dynamical systems exhibit non-predictability and non-controllability properties. Owing to exponential instability any attempt to predict their behaviour runs into practically insurmountable problems. That is the reason why, to forecast the evolution of such systems, one resorts to highly laborious methods employing more and more powerful computers. Recently several new methods for controlling chaotic systems have been proposed (Ott *et al* 1990, Ditto *et al* 1990, Hubler 1989, Jackson 1991, Basti *et al* 1991). The use some of them allows one to carry out stabilization of the unstable periodic orbits which exist in the chaotic attractor.

In the present study we considered another method—parametric destochastization—which allows one to easily realize control over dynamical systems with complex behaviour (i.e. to realize so-called non-feedback method for controlling chaos). With the use of weak, defined perturbations of some parameter one can cause the system with chaotic oscillations (that appear via destruction of the quasiperiodic motion) to transit into a regular regime. An important point is that such a regime is non-sensitive to small random external fluctuations. In turn, having got rid of the chaos, it is possible, in principle, to predict quantitatively dynamics of the systems. For systems in which development of chaotic behaviour may be described effectively by means of a quadratic map, the chaos can also be suppressed (Loskutov 1992a, b). Moreover, parametric destochastization is evidently an inherent part of self-organization phenomena when the order arises from the developed chaos. Although only the possibility of suppression of temporal chaos is studied, the appearance of parametric destochastization in distributed systems which can be approximated by a coupled-maps lattice is admitted as well (Loskutov and Thomas 1992).

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