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THERMODYNAMICS OF DISPERSING BILLIARDS WITH TIME-DEPENDENT BOUNDARIES

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By means of a thermodynamic approach we analyze billiards in the form of the Lorentz gas with the open horizon. For periodic and stochastic oscillations of the scatterers, the average velocity of the particle ensemble as a function of time is analytically obtained. It is shown that the consequence of such oscillations is Fermi acceleration which is larger for periodic oscillations. The described results do not depend on the size of scatterers and their position. Only the property of the horizon openness is necessary. It is found that the developed thermodynamic approach is in a very good agreement with the results of the direct numerical simulations at which the corresponding billiard map is used.

Keywords: Billiards; Lorentz gas; thermodynamics; Fermi acceleration; temperature.

1. Introduction

The classical billiard symbolizes a dynamical system where the point particle (billiard ball) reflects elastically from the boundaries of some closed region and performs free straight motion between reflections. Thus, the billiard particle moves along geodesic lines with a constant velocity. In the present paper we consider billiards in Euclidean plane. In this case, the angle of incidence of the particle is always equal to the angle of reflection.

A natural extension of billiard systems is to consider models for which the boundary is not fixed, but it oscillates with time. From the physical point of view such a generalized billiard consisting of many particles describes the rarefied gas in a container, the walls of which vibrate with a small amplitude. For billiards with the perturbed boundaries their dynamical properties are very essential. If it exhibits chaotic dynamics then perturbations of the boundary may lead to the infinite growth of the particle velocity. This problem is related to the unbounded increase of energy in periodically forced Hamiltonian systems and is known as the Fermi acceleration [Fermi, 1949; Ulam, 1961].

For the first time, to explain the origin of high-energy cosmic particles Fermi [1949] proposed the mechanism of the particle acceleration by means of collisions with moving massive scatterers. Later various models (the Fermi–Ulam model and others, see, e.g. [Brahic, 1971; Lichtenberg & Lieberman, 1990; Lichtenberg *et al.*, 1980; Pustyl'nikov, 1987, 1994, 1995; Pustyl'nikov *et al.*, 1995; Ulam, 1961; Zaslavsky, 1970]) have been developed which explained this phenomenon to a certain extent.

Billiards with perturbed boundaries can be considered as a certain generalization of such models. For example, in the papers [Koiller *et al.*, 1995; Koiller *et al.*, 1996] elliptic billiards have been numerically considered. The authors have come to the conclusion that, as in the Fermi–Ulam model, the growth of the particle velocity is bounded by invariant curves. Applying the dynamical approach (i.e. using the corresponding billiard dynamical systems) the Fermi acceleration has been analyzed on the example of the Lorentz gas and stadium-like billiards [Loskutov *et al.*, 1999, 2000; Loskutov & Ryabov, 2002]. Stadium-like billiards are defined as a closed domain with the boundary consisting of two focusing curves connected by the two parallel lines. It was found that in billiards with chaotic properties there is a linear growth of the particle velocity with time. However, for a nearly rectangle stadium with periodically oscillated boundary a new interesting phenomena is observed. Depending on the initial values, the particle ensemble can be accelerated, or its velocity will not grow, i.e. there is lack of the Fermi acceleration. This means that the boundary perturbation leads to the separation of the particle ensemble in velocities.

In these papers the following conjecture has been advanced: chaotic dynamics of a billiard with a fixed boundary is a sufficient condition for the Fermi acceleration in the system when a boundary perturbation is introduced. In papers [de Carvalho *et al.*, 2006; Karlis *et al.*, 2006] (see also references therein) the validity of this conjecture has been confirmed on the examples of some billiards of specific forms. Recently, more deep insight into the problem of the Fermi acceleration has been presented [Kamphorst *et al.*, 2007].

As it is known, the periodic Lorentz gas (with the fixed boundary) with the open horizon is a billiard which has strong chaotic properties (ergodicity, mixing, decay of correlations, etc.). This allows us to use a qualitative different method and apply a thermodynamic analysis to the system (see, e.g. [Kozlov, 2000, 2004; Chernov *et al.*, 1993; Bonetto *et al.*, 2002; Moran *et al.*, 1987]). Using such an approach one comes to the conclusion that *the obtained results should not depend on the billiard geometry*, i.e. on the size, curvature and dispositions of scatterers. The unique condition is the limitedness of the mean free path of the billiard particle, i.e. the Lorentz gas should possesses the bounded or the open horizon.

It should be noted that for the detailed analysis of such a complicated system as billiards it is necessary to use several approaches: dynamical, thermodynamical and computer simulations. The first method gives the exact description of some properties of the corresponding dynamical systems. The thermodynamical way, on the basis of a quite simple semi-phenomenological model, allows us to get the most general ideas about the system.

In the present paper, on the example of the Lorentz gas with the open horizon and time-dependent periodically and stochastically oscillating scatterers (i.e. the billiard boundary), we develop the thermodynamic approach to the description of such a system. We show that for both cases there is a Fermi acceleration. Thus, we support the above described conjecture advanced in papers [Loskutov *et al.*, 1999, 2000] about the Fermi acceleration in billiards with the perturbed boundaries.

2. The Lorentz Gas

A periodic Lorentz gas is a system containing a set of heavy discs (scatterers) with radius R embedded at sites of an infinite lattice with period a. Particles move freely among these discs. The billiard table in such a configuration is the whole plane except for the scatterers. Because particles do not interact with each other, then it is sufficient to consider only one particle. In Fig. 1 one can see a variant of the Lorentz gas in the periodic triangle lattice [Bunimovich & Sinai, 1981; Hakmi *et al.*, 1995; Machta & Zwanzig, 1983].

Depending on the radius R of the scatterers, this model possesses qualitatively different properties. If $R \leq a/2$ then the system has an infinite horizon. If $R \geq a/\sqrt{3}$, then the particle path is bounded by only one cell. When $a/2 < R < a/\sqrt{3}$ the free path of the particle is bounded, but it may move

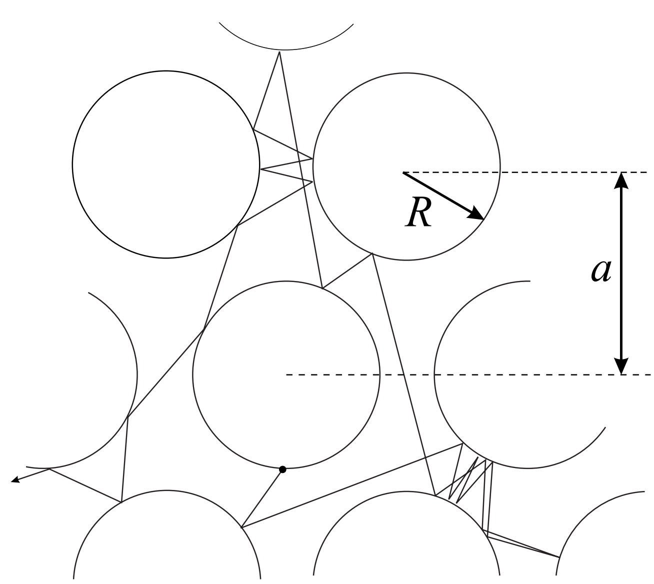

Fig. 1. Configuration of Lorentz gas model. The scatterers (circles of radius R) are located at sites of a lattice with period a.

freely in the billiard table. In this case the Lorentz gas has the open horizon.

For the infinite horizon, owing to the probability of long free paths, the statistical properties of a billiard are qualitatively changed. In particular, the mean free path does not converge, and there is an algebraic correlation decay with time [Baldwin, 1991; Bunimovich, 1985; Bunimovich & Sinai, 1981; Chernov, 1997; Garrido, 1997]. At the same time, for the Lorentz gas with the bounded horizon and the open horizon an exponential decay of correlations takes place, and the mean free path can be obtained as follows [Chernov, 1997]:

$$\lambda = \frac{\pi\Omega}{P}, \tag{1}$$

where Ω is the accessible billiard region and P is the scatterer perimeter.

Suppose that the radii of the scatterers are perturbed by a certain law, i.e. their boundaries are changed in such a way that $R(t) = R + r(t)$, where $\max|r(t)| \ll R$. We will analyze two different cases: stochastic and periodic (and in-phase) boundary oscillations. In the first case we assume that the phase of oscillations (and, as a consequence, the velocity of scatterer boundaries) at the collision time is a random value. The latter case corresponds to the situation when all the boundaries are perturbed by one and the same law and in the same phase. Physically this may be considered as an introduction of an external alternating field.

3. Stochastic Perturbations

To construct a thermodynamic model corresponding to a billiard with oscillating walls, it is logical to consider a number of identical particles with the mass m. In this case scatterers should be presented as heavy conglomerates consisting of the same rigidly connected particles. The boundary of such scatterers consists of N particles. Then collision of the moving particle with a scatterer may be described as the collision with one of the particles forming this scatterer. Also, this collision has a sense of the interaction of two thermodynamic subsystems (i.e. the moving particles and the fixed particles in scatterers) having their own temperatures. Then the interaction area S will correspond to a collision region and may be defined as the scatterer perimeter divided by N. It is obvious that the parameters m, S and N should not appear in the final result.

To compare our thermodynamical analysis with the known dynamical models [Loskutov *et al.*, 2000], we consider the Lorentz gas in a triangular lattice (Fig. 1). However, the proposed approach is much wider, and it can be applied to the description of the other types of billiards (see below).

It is well known that in the Lorentz gas with the open horizon the particle motion is stochastic. Therefore, the temperature of these particles can be associated with their average kinetic energy. The Hamiltonian of such a system corresponding to a single particle with the mass m is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + U(q_x, q_y),$$

where U is the energy of interaction with scatterers which fill area $L = \pi R^2$ such that $U(q_x, q_y \notin L) = 0$, $U(q_x, q_y \in L) = \infty$. For the two-dimensional motion, using the known result on the energy distribution into degrees of freedom, we get the expression for the particle energy:

$$\left\langle \frac{p^2}{2m} \right\rangle = \left\langle \frac{mv^2}{2} \right\rangle = kT. \tag{2}$$

By the same manner, the temperature of the scatterers can be determined by means of the kinetic energy of their motion, i.e.

$$\frac{M\langle u^2\rangle}{2} = \frac{Mu_0^2}{4} = kT_R. \tag{3}$$

At $u_0 \neq 0$ this temperature turns out to be infinite, $T_R = \infty$, because we consider the model with scatterers of the infinite mass, $M = mN \to \infty$. The conclusion about the infinite mass directly follows from the main assumption related to the Lorentz gas with the fixed scatterers: The angle of incidence is equal to the angle of reflection, i.e. the absence of recoil. Thus, the Fermi acceleration can be considered as a result of the heat exchange between the particle as a thermodynamic system and thermostat of the infinite temperature.

3.1. *The heat conductivity equation*

The heat flow from scatterers may be defined by the temperature gradient. Therefore, the heat conductivity equation can be written as follows:

$$\frac{\partial Q}{\partial t} = -\kappa S \frac{\partial T}{\partial x}, \tag{4}$$

where κ is a heat conductivity coefficient, S is a heat transfer area (the temperature gradient should be taken along the normal to this area). For the two-dimensional Lorentz gas this area S is defined as a collision area of the free particle and the scatterer boundary.

On the other hand, the heat flow can be expressed by the change of the particle energy:

$$\frac{\partial Q}{\partial t} = c\frac{\partial T}{\partial t} = \frac{c}{k}\frac{\partial}{\partial t}\left\langle \frac{mv^2}{2} \right\rangle, \tag{5}$$

where c is the heat capacity of the particle gas. In this case this parameter is equal to the Boltzmann constant k because the total input heat is transformed into the energy of the two-dimensional particle motion, i.e. into the temperature (see expression (2)). For such a process in the ideal gas the heat capacity is well known and defined by the expression $c = (i/2)kN_p$, where $N_p = 1$ is the number of moving particles, $i = 2$ is the number of degrees of freedom.

Let us find the explicit form of the right-hand side of Eq. (4). On the one hand, the effective temperature gradient between the particle and scatterers turns out to be infinite, $|\partial T/\partial x| \to \infty$, because $T_R = \infty$. However, on the other hand, as a consequence of point collisions the interaction area tends to zero, $S \to 0$. Thus, there is an indeterminate form $0 \cdot \infty$ which we can evaluate as follows.

It is obvious that $N \to \infty$. Then the interaction area corresponds to the particle size and can be written as

$$S = \frac{2\pi R}{N}. \tag{6}$$

The temperature gradient in Eq. (4) one can approximately write as $\partial T/\partial x \approx \Delta T/\Delta x$, where Δx is the distance where there is a temperature difference, and ΔT is the difference in the temperature between the scatterer and the particle gas. In turn, Δx we may approximately write as the free path: $\Delta x \approx \lambda_x = \lambda\langle v_x\rangle/\langle v\rangle = \pi\lambda/4$. For ΔT, taking into account the infinite temperature of scatterers, we get: $-\Delta T = T_R - T \approx T_R = mNu_0^2/4k$. Thus, evaluating an indeterminate form on the right-hand side of (4), finally we get:

$$-S\frac{\partial T}{\partial x} \approx \frac{2Ru_0^2 m}{\lambda k}. \tag{7}$$

The heat conductivity coefficient can be introduced in a usual manner:

$$\kappa = cDn, \tag{8}$$

where $n = 1/\Omega$ is a particle concentration, Ω is the area of the free space in the cell, D is two-dimensional diffusion coefficient along the normal to the scatterer boundary, c is the heat capacity. This diffusion coefficient can be written in the form

$$D = \frac{\langle v_x^2\rangle\tau}{2} = \frac{\langle v^2\rangle\tau}{3}, \tag{9}$$

where $\tau = \lambda/\langle v\rangle$ is the free path time. The free path λ may be presented as (see expression (1)):

$$\lambda = \frac{\Omega}{2R}. \tag{10}$$

Thus, the heat conductivity of the system particle-scatterers is determined as follows:

$$\kappa = \frac{1}{3}kn\langle v^2\rangle\tau. \tag{11}$$

By using Eqs. (7) and (11), the right-hand side of Eq. (4) is presented in the form:

$$\frac{\Delta Q}{\Delta t} = \frac{u_0^2 m\tau\langle v^2\rangle}{3\lambda^2}. \tag{12}$$

Now, combining the right-hand sides of Eqs. (5) and (12), we get:

$$\frac{d}{dt}\langle v^2\rangle = \frac{2u_0^2\tau}{3\lambda^2}\langle v^2\rangle.$$

Taking into account that for the ideal gas $\langle v^2\rangle = \alpha\langle v\rangle^2$, where α is a constant, we find the following expression for the change of the particle velocity in the Lorentz gas with the stochastically perturbed scatterer boundaries:

$$\frac{d}{dt}\langle v\rangle = \frac{u_0^2}{3\lambda}. \tag{13}$$

Thus, in the investigated system the Fermi acceleration is observed, and for the large enough particle velocity its average value grows as a linear function of time. It should be noted that the obtained result (13) is valid for the scattering billiard with an arbitrary shape of the boundary for which the free path is bounded.

4. Periodic Boundary Oscillations

For periodic oscillations of the boundary it is necessary to use another approach.

Let the boundary be perturbed in a regular way such that its velocity is $u = u_0 \cos\omega t$. In this case one can say that we do the work A on gas. Strictly speaking, the process is nonequilibrium, and the gas of billiard particles does not put pressure upon scatterers. However, we may consider a small enough area nearby the scatterer and admit that here the particle concentration n is constant. Then the pressure is determined as [Chichigina *et al.*, 2003]: $p(t) = mn(v+u_0 \cos \omega t)^2/2$. As a result of the scatterer motion the accessible billiard area Ω (see (1)) is changed. This change is defined by the scatterer velocity and the scatterer perimeter: $dV = 2\pi R u_0 \cos \omega t dt$. At that, the elementary work may be written as follows:

$$dA = pdV = \pi R u_0 mn(v + u_0 \cos \omega t)^2 \cos \omega t dt.$$

Assuming that the particle concentration in this chosen area is fixed and equal to $n = (2R\lambda)^{-1}$, we get: $dA = \pi u_0 m/2\lambda (v + u_0 \cos \omega t)^2 \cos \omega t dt$.

This work expends on the increase of the internal energy. Now, taking the average over the period of the scatterer oscillation and over the particle velocities, we obtain the differential equation which describes the change of the mean kinetic energy of the particle:

$$\frac{d}{dt}\left\langle \frac{mv^2}{2} \right\rangle = \frac{dA_T}{dt} = \frac{\pi m u_0^2 \langle v \rangle}{2\lambda}.$$

Owing to $\langle v^2 \rangle = \alpha \langle v \rangle^2$, where $\alpha \sim 1$, the Fermi acceleration for the case of periodic oscillations has the following form:

$$\frac{d}{dt}\langle v \rangle = \frac{\pi u_0^2}{2\lambda\alpha}. \tag{14}$$

Thus, periodic oscillations of the scatterers also lead to the Fermi acceleration. However, in the comparison with expression (13) the acceleration is higher for the stochastic boundary oscillations.

5. Numerical Simulations

In this part we present numerical simulations and compare them with the analytical results obtained above. All the calculations have been made by the dynamical approach, i.e. via the corresponding maps describing the particle motion in the Lorentz gas (Fig. 1) with the bounded horizon [Hakmi *et al.*, 1995; Loskutov *et al.*, 2000].

Consider the Lorentz gas model with the following parameters: the amplitude of the boundary oscillation of scatterers $u_0 = 0.01$, the scatterer radii $R = 0.56$, the cell size $a = 1$, the frequency of boundary oscillations $\omega = 1$, the initial velocity $v_0 = 1$. Thus, for the given billiard geometry the analytical value of the free path is $\lambda = 0.17$.

Numerical realizations of particle trajectories were different from each other in the initial directions of the particle velocity which were chosen in a random way. Two qualitative different cases were considered: stochastic oscillations of the scatterer boundaries with equidistributed phases and periodic oscillations. In both cases, the particle dynamics was determined by the map described in [Loskutov *et al.*, 2000].

In the stochastic case the oscillation velocity of the boundary was defined as follows: $u_n = u_0 \cos \phi_n$, where ϕ_n is a uniformly distributed random value over the interval $[0, 2\pi]$. For periodic oscillations $u_n = u_0 \cos \omega t_n$, where t_n is the instant of the nth collision with the boundary. For each case, 5000 realizations of the trajectory of the billiard particle have been constructed. Dynamics of the particle ensemble was simulated up to $3 \cdot 10^5$ time units, and some trajectories (of the particles with high velocities) include up to 10^7 iterations.

The averaged dependencies of the particle velocity for the stochastic v_{st} and periodic v_{reg} boundary perturbations are shown in Fig. 2. In this figure full lines correspond to the regular case, and dotted lines correspond to the stochastic case. Straight lines are analytical results (13) and (14). As follows from this figure, the acceleration has a linear character that is in good agreement with the obtained analytical expressions. The results are the same for other parameters of the Lorentz gas with the open horizon.

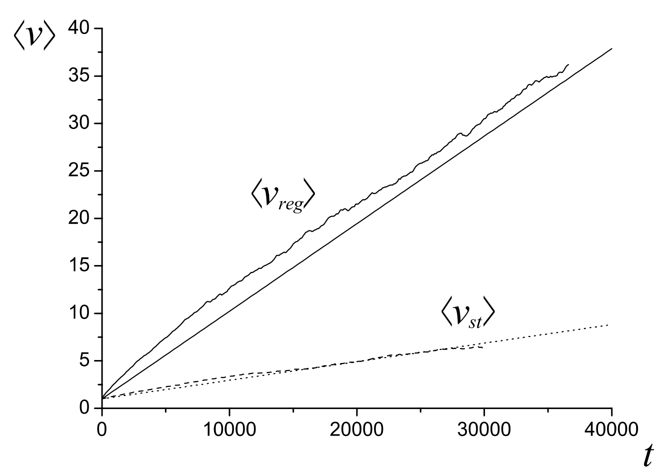

Fig. 2. Average particle velocities as functions of time in the Lorentz gas. The following parameters were used: $u_0 = 0.01, a = 1, R = 0.56$ and $v_0 = 1$. The average curve was obtained on the basis of 5000 realizations of different directions of the initial velocity v_0, which were selected in a random way.

6. Conclusion

The Lorentz gas is a widely recognized model of the ideal rarefied gas. For the Lorentz gas with the open horizon a number of its statistical properties is proved: mixing, exponential decay of correlations, the existence of the diffusion coefficient [Bunimovich & Sinai, 1981]. Thus, for the analysis of the particle motion in this system one can use the thermodynamical approach. In a certain sense, such an approach is a universal one because in this case the geometry (i.e. the disposition of the scatterers, their forms, sizes and curvature) does not play an essential role. A quite natural generalization of the classical Lorentz gas is the system with oscillating scatterers.

In the present paper, on the basis of the thermodynamical analysis we have shown that at the scatterer perturbations of the Lorentz gas with the open horizon the average velocity of the particle ensemble (for both, stochastic and periodic oscillations) grows linearly with time (see (13), (14)), i.e. the Fermi acceleration phenomenon is observed. This confirms the conjecture advanced by the authors of the papers [Loskutov *et al.*, 1999, 2000] that chaotic dynamics of a billiard with a fixed boundary is a sufficient condition for the Fermi acceleration in the system when a boundary perturbation is introduced. It is also found and explained that the acceleration is higher in the case of periodic oscillations than for the stochastic perturbations.

The model proposed in the present paper and based on the assumption that the scatterers consist of an infinite number of particles of one and the same kind, is not uniquely possible. But, nevertheless, the results obtained on its basis are in a very good agreement with the conclusions which follow directly from the dynamical theory of the lattice Lorentz gas (see [Bunimovich & Sinai, 1981; Loskutov *et al.*, 1999, 2000]). Also, using this model one can obtain the value of the Fermi acceleration for the arbitrary distributed scatterers, scatterers of the arbitrary form and the motion, and for three-dimensional Lorentz gas. The unique claim here is the condition of the open horizon. In addition, on the basis of the proposed model one can easily explain the difference in the results for the regular and stochastic motions of the scatterers.

Namely, the Fermi acceleration is a dynamical effect that appeared in the time-dependent Lorentz gas. It is known that the transformation of the heat energy into a mechanical form is restricted by the second law of thermodynamics. In the case of stochastic oscillations we have exactly this type of the transformation. At the same time, the axiomatic notion of the work (i.e. a mechanical form of the energy transfer) used for the case of periodic scatterer oscillations, implies a straight energy transfer from scatterers to moving particles.

Finally, we should emphasize that the principal result of our paper is not the analytical expressions for the Fermi acceleration. They are necessary for the verification of the model premise. The main conclusion is that our approach allows to get results for a quite wide class of the parameters of the Lorentz gas.

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