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STABILIZATION OF CHAOTIC BEHAVIOR IN THE RESTRICTED THREE-BODY PROBLEM

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The restricted three-body problem on the example of a perturbed Sitnikov case is considered. On the basis of the Melnikov method we study a possibility to stabilize the obtained chaotic solutions by two bodies placed in the triangular Lagrange points. It is shown that in this case, in addition to regular and chaotic solutions, there exist stabilized solutions.

Keywords: Restricted three-body problem; Sitnikov problem; Melnikov method.

1. Introduction

Henri Poincaré in his work on Celestial Mechanics underlined a possibility of chaotic behavior in three-body problem by the destruction of homoclinic contours. Later, an existence of transverse homoclinic points in the three-body problem was analytically verified. A well-known modification of the restricted three-body problem is that of Sitnikov [1960].

The Sitnikov problem takes place when two equal masses M orbit around their barycentre. A third, massless or small but finite mass body (particle) μ moves in their gravitational field perpendicular to the motion surface of the primaries (see Fig. 1). It can be shown, that the oscillation of the third body is chaotic (under certain additional conditions). In this work we consider the problem of stabilization of this chaotic behavior. In general, this problem is related to stabilization and control of unstable and chaotic behavior of dynamical systems by external forces. A comprehensive study of chaotic systems with external controls may thus provide a key to the understanding many nonlinear processes in both localized and distributed systems. This could be of interest, for example, in the study of planetary systems of binary stars.

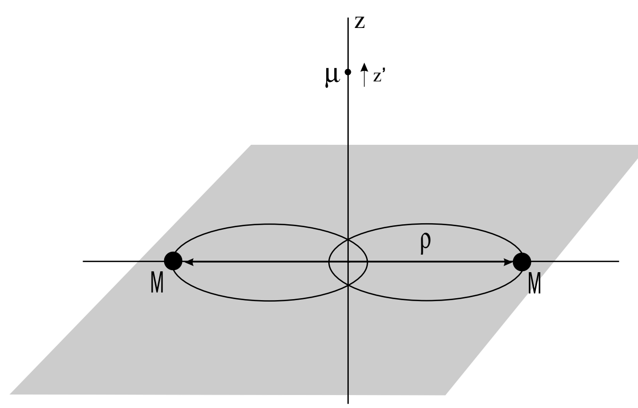

Fig. 1. The Sitnikov problem.

2. Sitnikov Problem

With the proper time and space scaling we can write the corresponding differential equation for the particle μ in the form

$$\ddot{z} = -\frac{z}{(z^2 + \rho^2)^{3/2}}, \tag{1}$$

where $\rho = \rho(t) = 1 + \varepsilon \cos t + O(\varepsilon^2)$. Here the small parameter ε is closely related to the eccentricity. To make clear how this problem is related to homoclinical orbits, we introduce noncanonical transformation [Dankovicz & Holmes, 1995]: $z = \tan u$, $v = \dot{z}$, $u \in [-\pi/2, \pi/2]$, $v \in R$. Then the Hamiltonian for Eq. (1) in the new variables (u, v) has the form:

$$\begin{aligned} H(u,v) &= \frac{1}{2}v^2 - \frac{1}{\tan^2 u + \rho^2} \\ &= H_0(u,v) + H_1(u,v,t,\varepsilon), \end{aligned} \tag{2}$$

where $H_0(u,v) = 1/2v^2 - \cos u$. As we can see, when $\varepsilon = 0$ our system reveals the dynamics of a nonlinear pendulum, which oscillates with increasing amplitude in time. If $\varepsilon \neq 0$ then the system (1) exhibits chaotic properties [Dankovicz & Holmes, 1995].

3. Stabilization of Chaotic Behavior

As follows from the work of V. M. Alekseev [2001] one can find one-to-one correspondence between the collinear solutions set of the Sitnikov problem and the symbolic set Ω: $v^- i_{n_1} | \ldots m_{n-1} i_{n-1} m_n i_n m_{n+1} i_{n+1} \ldots | v^+$, where m_{n_1}, $n_1 < n < n_2$ are natural numbers $\geq N$, $i_n = 0$ or 1, $v^\pm \in [0, \delta]$, $n_1 < 0 \leq n_2$. The number of collinear configurations on the solution set is defined by i_n. Symbol m_n indicates the number of total rotations of the mass M bodies between $(n-1)$-th and nth system collinear configuration. Therefore, we may choose m_n such that our system oscillates slowly nearby its origin (when $\varepsilon = 0$). At $\varepsilon \neq 0$ (a perturbed chaotic nonlinear pendulum) there are transverse homoclinic points.

To clarify the sense of this statement, let us consider a symbolic sequence [Alekseev, 2001]:

$$\nu^- \underbrace{\overset{\uparrow}{0} N_1 \overset{\uparrow}{1} N_1 0 \ldots \overset{\uparrow}{1}}_{2k_1 \text{ times}} N_1 \underbrace{\overset{\uparrow}{0} N_2 0 \ldots N_2 \overset{\uparrow}{0}}_{2k_2 \text{ times}} N_1 1 N_1 0 N_1 1 \nu^+$$

This sequence can be interpreted in the following manner. A spacecraft μ which arrived at a binary star system from infinity with velocity ν^-, first came to the nearly periodic orbit with period $4\pi N_1$. In this orbit, it made k_1 complete oscillations. During each of these oscillations, the spacecraft returned twice to the mass centre of the binary star in the moments of maximal and minimal distances between the components of the binary star. Then the spacecraft moved to another nearly periodic orbit with period $4\pi N_2$. Here it made k_2 complete oscillations. Each of the $2k_2$ spacecraft returns to the mass centre takes place in moments of the minimal distance between the components of the binary star. Finally, having returned to the initial orbit, the spacecraft μ made one and half oscillation near this orbit. Then it moved away to infinity with velocity ν^+ following the same direction from which it came initially.

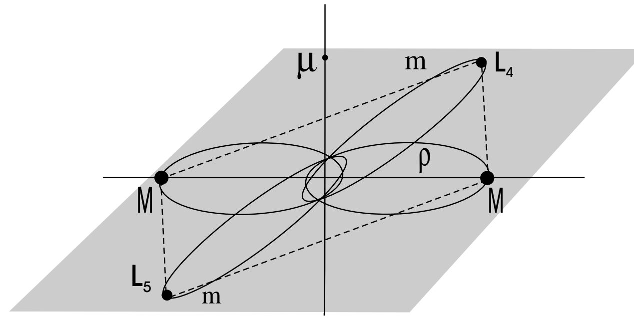

Fig. 2. Stabilization of particle's chaotic behavior.

Now, placing two new bodies of the mass m (here $M \gg m \gg \mu$) in the Lagrange points L_4 and L_5 we can achieve the situation when the influence of these bodies on the mass μ has a form of periodical (forced) impulses (the trajectories of bodies of mass m are shown in Fig. 2). Therefore, this may be treated as a nonlinear perturbed pendulum with a specific external force.

Earlier it has been shown [Loskutov & Dzhanoev, 2004] that using Melnikov method, we can obtain the stabilized dynamics for μ. In this case, duration time τ of these impulses should be much less than the characteristic time T of the system, i.e. $\tau \ll T$ and $\tau \to 0$. Thus, the stabilized system has the following form:

$$\begin{aligned} H(u,v) &= H_0(u,v) \\ &\quad + \varepsilon \left[H_1(u,v,t) + \sum_n \delta(t - n\tau) \right]. \end{aligned} \tag{3}$$

Between pulses the motion of the particle μ is free. Then, speaking in terms of celestial mechanics, our system undergoes the influence of some exterior celestial bodies (say, two spacecrafts) which orbit near the third body (particle).

4. Numerical Results

The onset of chaos in dynamics of the mass μ in three-body system (Fig. 1) corresponds to the breakdown of a heteroclinic trajectory. Figure 3 illustrates the structure of a typical chaotic set obtained in this case. In Fig. 4, numerical solutions of the system with Hamiltonian (3) is shown. It is clear that the dynamics of the particle μ tends to a regular regime represented by a periodic orbit.

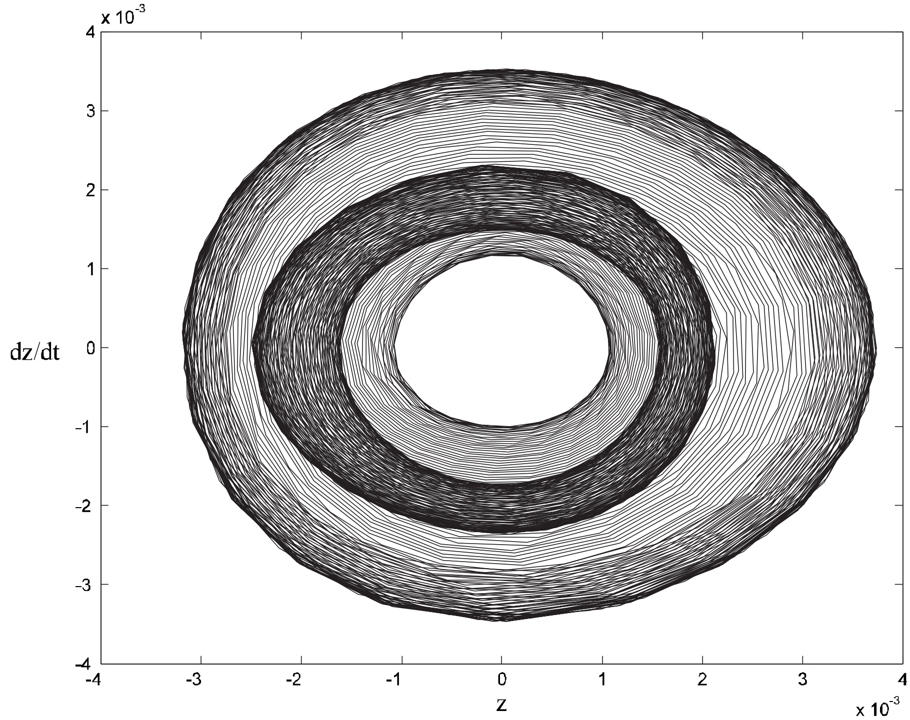

Fig. 3. Numerical solution to system with Hamiltonian (2).

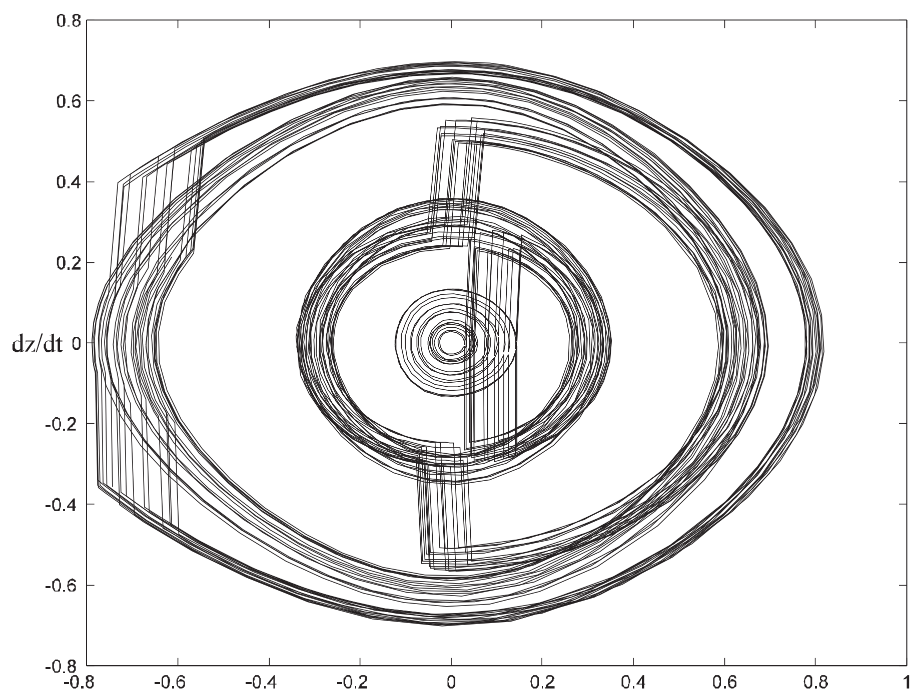

Fig. 4. Numerical solution to system with Hamiltonian (3).

The analysis of invariant characteristics of chaoticity (Lyapunov exponents, power spectra, etc.) confirms this conclusion.

5. Concluding Remarks

Separatrix splitting is a very convenient method for examining dynamical systems behavior, because it can be used to obtain nonintegrability conditions for many applied problems in an analytical form. As a result, the distance between the splitting separatrices can be found by applying a perturbation method in the vicinity of a homoclinic trajectory. In this study, separatrix splitting is applied to explore the possibility of chaos suppression (stabilization) [Alekseev & Loskutov, 1987; Ott *et al.*, 1990] in the perturbed restricted three-body problem (so-called Sintikov case). On the basis of the Melnikov method, it is found that stabilization of chaotic solutions can be obtained by placing two bodies in the triangular Lagrange points. It is shown that in this case, in addition to regular and chaotic solutions, there exist zones of the stabilized behavior.

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