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Inducing Stable Periodic Behaviour in a Class of Dynamical Systems by Parametric Perturbations

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Abstract— We develop an analytic approach to the problem of the behaviour stabilization of dynamical systems by parametric perturbations. First, questions of realization of the stable dynamics in non-chaotic systems with continuous time are studied. It is analytically shown that by means of parametric perturbations it is possible to obtain the stable periodic behaviour in systems which do not possess stable oscillations in the autonomous case. Then, on the basis of these results we advance the following conjecture: assume that a system has a chaotic attractor. Then, if we successfully choose a parametric perturbation of such a system in those regions where its behaviour is chaotic, then one can expect that this perturbation leads to the appearance of the stable periodic orbits which are either unstable or non-existent in the initial unperturbed system. We present rigorous results which assert that for some discrete time dynamical systems such a conjecture is valid: for certain families of mappings it is possible to find perturbations that lead to stabilization of their chaotic dynamics. In the framework of such an approach we offer a goal-oriented non-feedback way for stabilization of the desired stable periodic behaviour. Copyright ©1996 Elsevier Science Ltd.

1. INTRODUCTION

Nowadays a problem related to controlling the evolution of dynamical systems with complex behaviour attracts the attention of many authors (see, for example, [1], [2] and [3] and references therein). In such investigations the form of dynamical systems is as follows

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, \alpha), \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^n$, $\alpha \in A$ and the components α_i of the vector α are parameters. Without loss of generality we admit that only one parameter, $\alpha_k \equiv \alpha$, can be varied. Let us suppose that for any admissible values of parameters from A the system dynamics is not acceptable with the applied point of view. For example, if for all possible $\alpha \in A_c \subset A$ system (1) possesses chaotic attractors (i.e. prediction of its evolution is difficult) or the function $\mathbf{v}(\mathbf{x}, \alpha)$ does not have stable critical elements, then for ecological, chemical and some other systems it is required to avoid similar behaviour. In these cases one should find a certain control action leading to appearance the desired dynamics. In other words, it is necessary to realize the control of the system dynamics through the transformation of the function $\mathbf{v}(\mathbf{x}, \alpha)$ such that a new system

$$\dot{\mathbf{x}} = \mathbf{v}'(\mathbf{x}, \alpha, t) \tag{2}$$

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would be characterized by the required behaviour.

In the control theory two methods of transformation of the right-hand side of system (1) are known: (1) the open-loop control and (2) the feedback control. In the first case the control is realized by (a) force perturbation, $\mathbf{v}'(\mathbf{x}, \alpha, t) = \mathbf{v}(\mathbf{x}, \alpha) + \mathbf{g}(t)$, and (b) parametric perturbation, $\mathbf{v}'(\mathbf{x}, \alpha, t) = \mathbf{v}(\mathbf{x}, \alpha_0 + \alpha_1(t))$. For periodic perturbations, $\mathbf{g}(t)$ and $\alpha_1(t)$ are T-periodic functions, $\mathbf{g}(t+T) = \mathbf{g}(t)$, $\alpha_1(t+T) = \alpha_1(t)$.

In the second case an external force and the parametric control action are functions of the dynamical variables, i.e. (a) $\mathbf{v}'(\mathbf{x}, \alpha, t) = \mathbf{v}(\mathbf{x}, \alpha) + \mathbf{g}(\mathbf{x}(\tau))$ and (b) $\mathbf{v}'(\mathbf{x}, \alpha, t) = \mathbf{v}(\mathbf{x}, \alpha(\mathbf{x}(\tau)))$. Moreover, for the feedback control the change in the external forcing or the control parameters depends either on the current system state, $\tau = t$ (i.e. the *momentary feedback* is realized), or on its previous state, $\tau > t$ (i.e. the *delayed feedback* is established).

For chaotic systems the feedback parametric method has been proposed by Ott *et al.* [4] and later developed by many authors (see, for example [5], [6] and also [2] and references therein). The non-feedback parametric way of stabilization of the chaotic behaviour (in fact, chaos suppression) in dynamical systems has been apparently proposed by Alekseev and Loskutov [7,8]. Somewhat different approaches have been investigated by Lima and Pettini [9] and Kivshar *et al.* [10]. The original goal-oriented methods of controlling chaotic behaviour have been described by Hübler [11], Lüsher and Hübler [12], Jackson and Hübler [13,14] and by Pyragas [15,16].

Unfortunately, for certain systems (for example, chemical and biological) realization of the forcing control is quite difficult in applications (if it is possible at all). Often in such systems the dynamical variables x_i are proportional either to the total mass or the relative mass of the reacting substances. Then $\mathbf{v}(0, \alpha) = 0$, and the hypersurfaces $x_i = 0$ are invariant. With such a setting up problem, system (1) only describes the real processes in the region $\mathbf{G} = \{\mathbf{x} \mid x_i \geq 0, \sum_i x_i \leq M\}$, $M = const$. In this case the force action can lead to the fact that phase trajectories can leave the region $\mathbf{G}$ crossing the hypersurface $x_k = 0$. Therefore the parametric control is only admissible here. In turn, the feedback method requires specification of the system state, which is quite difficult in certain cases. This is why it is desirable to exclude feedback. Although such a method seems to be more simple than others, a satisfactory theory describing the possibility of obtaining the desired behaviour of the chaotic system and an explicit form of the periodic alteration of the parameters is absent.

In the present paper, for some classes of dynamical systems we analytically investigate the problem of the behaviour stabilization by means of parametric perturbations and thus, we hope to make up for a deficiency in the theory. Finally, we offer use of our feedback-free method for stabilization of the desired periodic orbits and therefore, to realize the goal-oriented control of dynamical systems.

2. PERTURBATIONS OF NON-CHAOTIC DYNAMICAL SYSTEMS

Let us consider some assertions concerning the parametric action on the systems which do not have stable periodic behaviour. We study the following two systems of equations,

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= \alpha y[x^4(1+2a) - \frac{1}{8}(1-a)] - x\,, \end{aligned} \tag{3}$$

and

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\alpha y(x^2 + ax + 1) - x\,, \end{aligned} \tag{4}$$

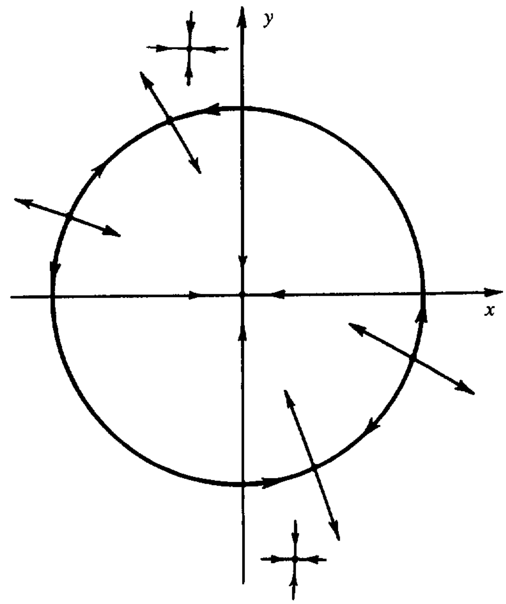

Fig. 1. The Poincaré map $\tau = 0$ for system (5) at $h = (\max(h_1, h_2), h_3)$.

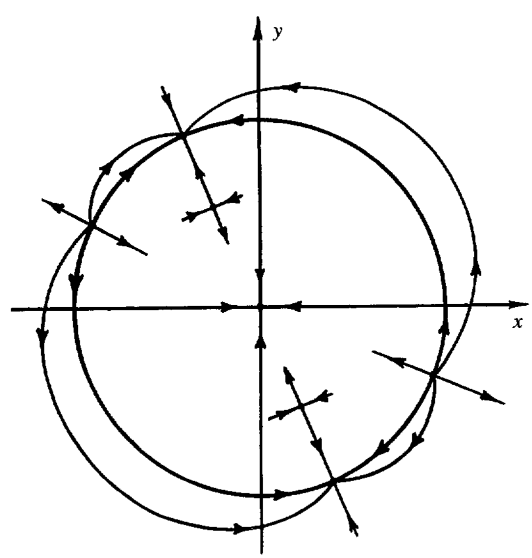

Fig. 2. The Poincaré map $\tau = 0$ for system (5) at $h \in (h_3, h_4)$.

in some bounded domain U_0, where α satisfies the inequality $0 \leq \alpha \ll 1$, and a is a control parameter. These systems are equivalent to the system of the Van der Pol type; similar systems are often used as a mathematical model of certain non-linear radio physical generators [17].

First let us dwell on system (3). The structure of the domain U_0 in (3) is sufficiently simple. To wit,

(a) for $a < -1/2$ system (3) has the unique stable focus;

(b) for $a \in (-1/2, 1)$ except for a stable focus, in (3) there exists an unstable periodic orbit. Note

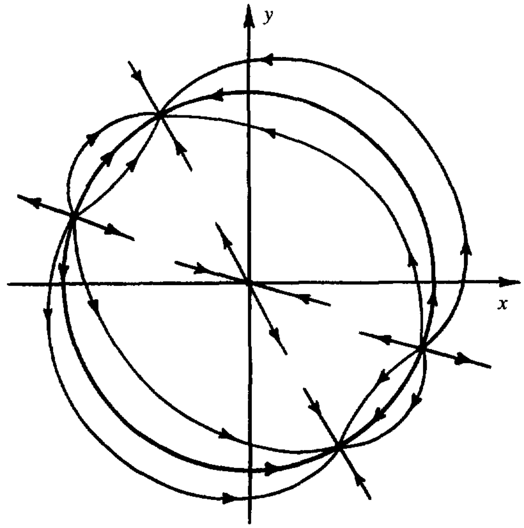

Fig. 3. The Poincaré map $\tau = 0$ for system (5) at $h > h_4$.

that in a zero approximation in α, the periodic orbit has the radius $r = [(1-a)/(1+2a)]^{1/4}$ and therefore for $a \to -1/2$ this periodic orbit cannot lie in the domain U_0;

(c) for $a > 1$ the domain U_0 contains only one unstable focus.

Thus, system (3) does not possess non-trivial stable periodic orbits for any bounded a. Let us introduce the parametric perturbation of period $T = 2\pi/\omega$ in (3) in the following manner:

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= \alpha y[x^4(1 + 2h\cos 2\omega\tau) - \frac{1}{8}(1 - h\cos 2\omega\tau)] - x\,, \\ \dot{\tau} &= 1\,, \end{aligned} \tag{5}$$

where h is an amplitude of perturbations, $\omega = 1/(1+\alpha\eta)^{1/2}$ and $\eta > 0$ is a constant. Equations (5) are determined in a bounded domain $U = U_0 \times \mathbf{R}/T\mathbf{Z}$ containing the origin, where U_0 is a domain of the Euclidean space $\mathbf{R}^n$. Also, system (5) can have in U a so-called *trivial* periodic orbit L_0^T, i.e. a set $0 \times \mathbf{R}/T\mathbf{Z}$. Now, going to the truncated system by means of a change of variables, $\theta = \omega\tau$, $x = b\cos(\varphi + \theta)$ and the average procedure, we get

$$\begin{aligned} \frac{\mathrm{d}b}{\mathrm{d}\theta} &= \alpha B(b,\varphi) = \alpha\frac{b}{16}(b^4 - 1)(1 + \frac{h}{2}\cos 2\varphi)\,, \\ \frac{d\varphi}{d\theta} &= \alpha\Phi(b,\varphi) = \alpha[\frac{\eta}{2} + \frac{h}{32}(5b^4 + 1)\sin 2\varphi]\,. \end{aligned} \tag{6}$$

It is known [18] that the steady-state solution b_0, φ_0, that is

$$B(b_0,\varphi_0) = \Phi(b_0,\varphi_0) = 0, \quad \left.\frac{\partial(B,\Phi)}{\partial(b,\varphi)}\right|_{b=b_0,\varphi=\varphi_0} \neq 0, \tag{7}$$

corresponds to the periodic orbits of system (5) in the order zero of the perturbation theory, and the stability of these orbits agree with the stability of solution (7). Moreover, the bifurcation values of the parameter h in (6) are coincident with the corresponding values for system (5) up to $O(\alpha)$. Thus, by analytical methods one can find a qualitative modification of the system dynamics when the amplitude h increases. We illustrate this phenomena by Poincaré mapping $\tau = 0$. Namely,

(1) At $h = 0$, system (5) possesses the trivial stable periodic orbit L_0^T and an unstable invariant torus Tor^2.
(2) At $h \in (0, \min(2, 8\eta/3))$ on the torus Tor^2 saddle unstable periodic orbits can appear with periods more than T.
(3) If $h = h_1 = 2 + \delta(U)$ (where $\delta(U)$ is introduced in connection with the finiteness of U_0) then, except for L_0^T and Tor^2, there are two stable periodic orbits L_1^T and L_2^T of the period T in the domain U (Fig. 1). With increasing h the periodic orbits L_1^T, L_2^T are monotonically pulled to L_0^T.
(4) For $h = h_2 = 8\eta/3$ on the torus Tor^2, two pairs of the periodic orbits appear: two of them are saddle, L_3^T and L_4^T, and two are unstable, L_5^T and L_6^T. Note, that if $\eta < 3/4$ then items (3) and (4) are interchanged.
(5) When $h = h_3 = 2[1 + (4\eta/3)^2]^{1/2}$, confluence of the stable periodic orbits L_1^T and L_2^T with the saddle ones L_3^T and L_4^T, respectively, takes place, so that the periodic orbits L_3^T and L_4^T become stable (Fig. 2), and the orbits L_1^T and L_2^T become saddle.
(6) For $h = h_4 = 2[1 + (8\eta)^2]^{1/2}$, the orbits L_1^T, L_2^T and L_0^T stick together. In this case the trivial periodic orbit L_0^T becomes saddle.
(7) For $h > h_4$ in system (5) there are the saddle trivial periodic orbit L_0^T, the stable periodic orbits L_3^T and L_4^T and the unstable periodic orbits L_5^T and L_6^T (Fig. 3).

Let us now consider system (4). It is easy to come to a conclusion that for any bounded parameter value of a, this system has only the unique stable focus. Introducing a parametric perturbation as follows:

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\alpha y[x^2 + (1 + h\cos 3\omega\tau)x + 1]\,, \\ \dot{\tau} &= 1\,. \end{aligned} \tag{8}$$

For this system the trivial periodic orbit L_0^T is always stable and for $h^2 < h_1^2 = 8[1 + (1 + \eta^2)^{1/2}]$ there are no the other stable trajectories in U. At $h^2 = h_1^2$ bifurcation of the birth of three pairs of periodic orbits takes place: three of them are saddle and the others are stable. In the Poincaré mapping $\tau = 0$, this gives the appearance of three saddle-nodes; each of these saddle-nodes is then decomposed into a saddle and a stable node. The distance ρ from these orbits to the trivial state can be calculated as follows:

$$\rho_{sad}^2 = \frac{h^2 - 8 - \sqrt{h^4 - 16h^2 - 64\eta^2}}{2} + O(\alpha)\,,$$
$$\rho_{st}^2 = \frac{h^2 - 8 + \sqrt{h^4 - 16h^2 - 64\eta^2}}{2} + O(\alpha)\,.$$

Thus, in the parametrically perturbed system (4) there are three non-trivial stable periodic orbits.

Remark 1. It should be noted that in view of the small parameter α the nearer the modulus of the multiplicator of the unstable periodic orbit to 1, the smaller the amplitude of the parametric perturbation (required for the appearance of stable periodic behaviour) can be.

Remark 2. It is easy to obtain a similar result for the systems of higher dimensions. For example, for systems which can be described by a direct product of (5) or (8) and the equations of the form $\dot{\mathrm{y}} = \Lambda \mathrm{y}$, where Λ is a matrix having the eigenvalues with negative real parts, there is a parametric perturbation leading to the appearance of stable periodic orbits.

Remark 3. In the case $\alpha \to 0$ the distance ρ does not tend to zero. This means that for a small enough α, stable periodic solutions have a finite amplitude.

Thus, for a certain class of dynamical systems which in autonomous cases do not possess stable periodic orbits, it is possible to find the stabilizing parametric perturbations.

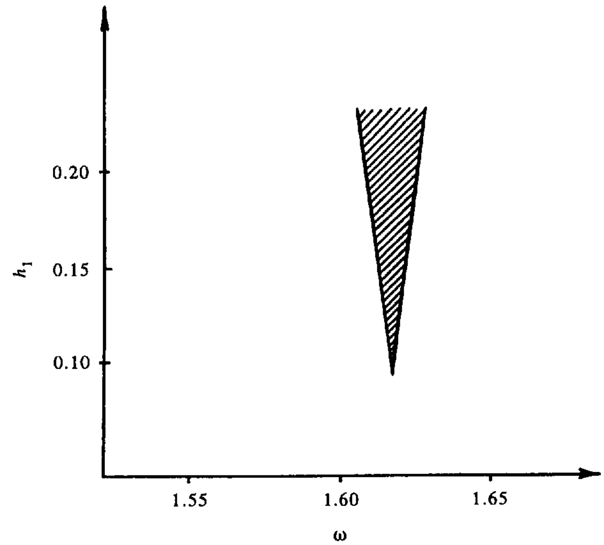

Fig. 4. The region of parametrically stabilized behaviour in the Rössler system (9).

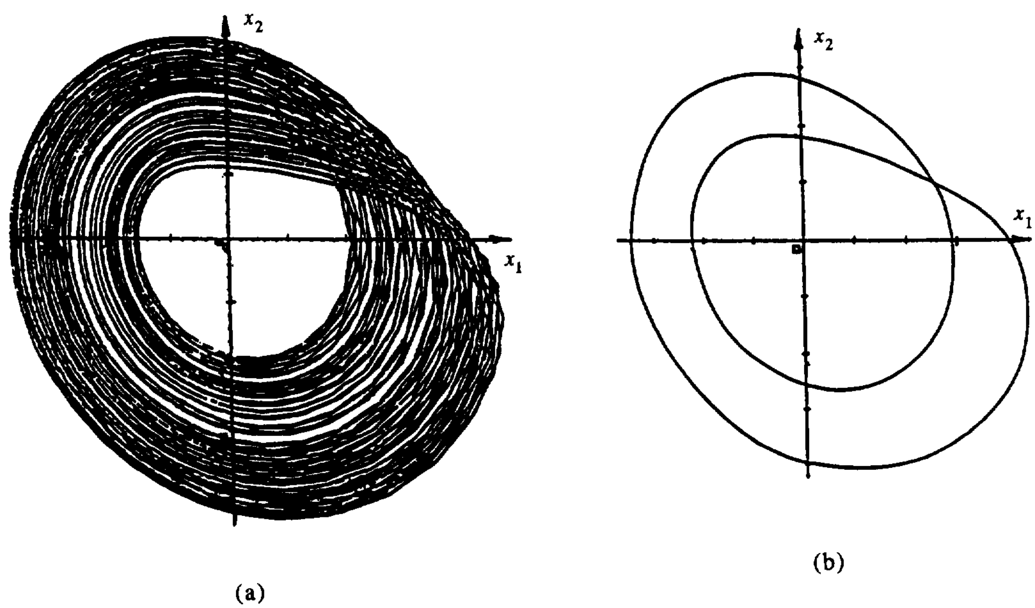

Fig. 5. A chaotic trajectory of system (9) at $a = 4.46$ (a) and the stabilized periodic orbit at $h_0 = 4.46$, $h_1 = 0.21$ and $\omega = 1.62106$ (b).

3. PERIODIC PERTURBATIONS OF CHAOTIC DYNAMICAL SYSTEMS

The results obtained in the previous section allow us to advance the following conjecture:

- Suppose that a system possesses a chaotic attractor. As is known, such types of attractors contain a set of saddle periodic orbits. Then, if we can correctly find the frequency of the parametric perturbation, then for certain amplitudes it is possible to obtain the stable

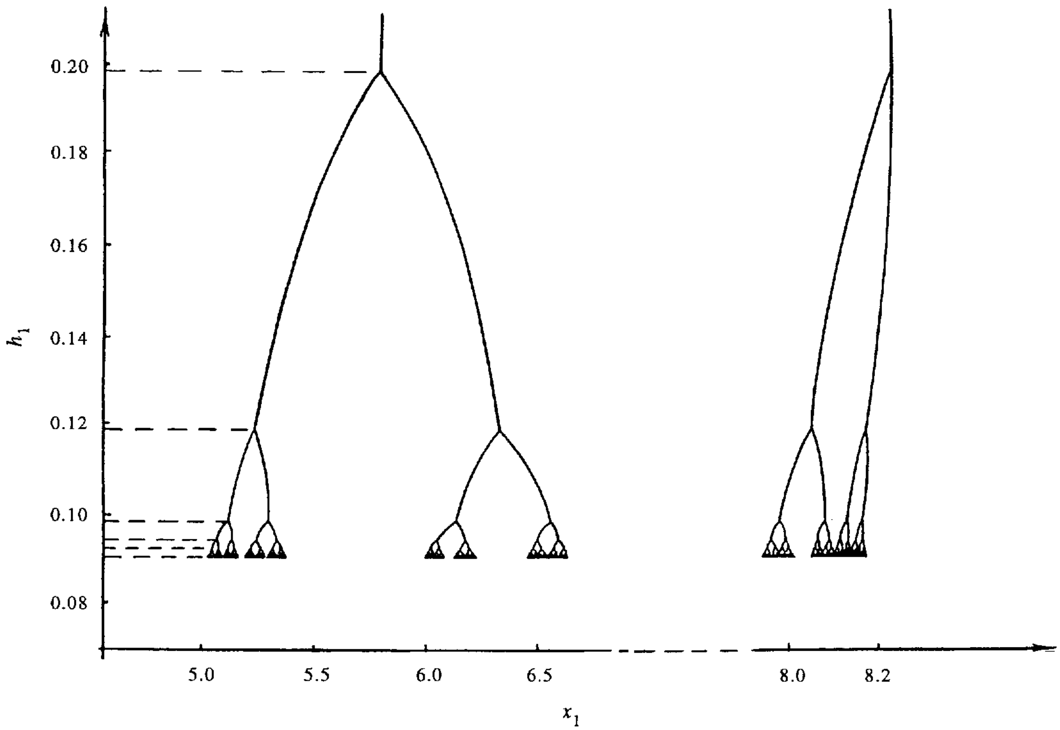

Fig. 6. The bifurcation diagram of the stabilized periodic orbits in the perturbed Rössler system (9) at the variation of the amplitude h_1.

periodic motion which either did not exist in the initially unperturbed chaotic system or was unstable.

This section is devoted to the substantiation of this conjecture. First let us describe one of the numerical results concerning the stabilization of chaotic behaviour [8]. Consider as an example a non-linear system of the following differential equations:

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 , \\ \dot{x}_2 &= x_1 + x_2/5 , \\ \dot{x}_3 &= -ax_3 + x_1x_3 + 1/5 , \end{aligned} \tag{9}$$

which is known as a Rössler system [19]. System (9) was used as the simplest mathematical model of some oscillatory chemical reactions [20]. Let us introduce the parametric perturbation as follows: $a \to h_0 + h_1 \sin \omega t$, where $h_0 = (a'' + a')/2$, $h_1 \leq (a'' - a')/2$, and a'', a' are the boundary points of one of the chaotic regions A_c in system (9). The latter assertions are necessary to guarantee that the perturbed Rössler system remains within A_c. By numerical investigations one can easily find that the chaotic set A_c appears via period doubling bifurcations and $A_c = (a'', a') = (4.22, 4.69)$. Carefully selecting the frequencies ω of the perturbation we can find the region of stabilized behaviour of system (9) (Fig. 4). In this region it is possible to stabilize the unstable periodic orbits embedded in the chaotic attractor and thus, to realize the goal-oriented control without feedback [Fig. 5(a) and(b)]. This is due to a special form of the right-hand side of the Rössler equations (9) [31]. Varying the amplitude h_1 or the frequency of the perturbation we can get the stabilized periodic orbits (Fig. 6) of one or another period. It should be noted that the amplitude of perturbation $h_1 \ll h_0$, i.e. the obtained controlling, should be realized by a small enough perturbation.

Thus, to realize a periodic behaviour in some chaotic flows it is sufficient to apply a weak parametric perturbation within the chaoticity region. However, it would be much more interesting to develop an analytic approach to this problem. The rest of this section is devoted to description of such an approach. We rigorously show that for certain maps it is possible to find external parametric perturbations leading to stabilization of the behaviour. However, first of all, let us dwell on the general properties of maps with external perturbations.

Usually n-dimensional maps have the form

$$T_a : \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, a) , \tag{10}$$

where $\mathbf{f} = \{f_1, \ldots , f_n\}$ is a certain function, $\mathbf{x} = \{x_1, \ldots , x_n\}$ and $a \in A$ is a control parameter. Let us introduce a parametric perturbation $G : A \to A$ of map (10) as follows:

$$G : a \mapsto g(a), \ a \in A. \tag{11}$$

Let us confine ourselves by the periodic (i.e. cyclic in the parameter a) perturbation with period $\tau : \ a_{i+1} = g(a_i), \ i = 1, 2, \ldots , \tau - 1, \ a_1 = g(a_\tau), \ a_i \neq a_j$ for $i \neq j, \ a_i \in A, \ 1 \leq i \leq \tau$. In this case we get the following result:

Proposition 1 ([22,23]). The period of an obtained periodic orbit in the periodically perturbed map (10) is always multiple to the period of the perturbation.

However, the following remark should be noted here. Introduction of a perturbation means that we consider an $(n + 1)$-dimensional map:

$$\mathbf{T} = \begin{cases} \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, a) , & \\ a \mapsto g(a) , & a \in A, \end{cases} \tag{12}$$

and a projection of this map onto the coordinate plane $(x_1, \ldots , x_n)$ is the perturbed map (10). In general, the projection of the period t orbit onto the coordinate plane is also an orbit of period t. However, at this type of projection specific (degenerated) cases may occur, when two or several points are projected onto one and the same points in $(x_1, \ldots , x_n)$. Then the representative point of the perturbed map hits at several points forming a periodic orbit twice or several times. For example, for $t = 2$ it is possible to observe only one point in the projection $(x_1, \ldots , x_n)$. Naturally, such specific periodic orbits cannot be called periodic orbits in the usual sense.

Now we consider the following useful construction which can facilitate the study of perturbed maps. τ-cyclic transformation (11) means that the perturbed maps can be rewritten as follows:

$$\mathbf{T} = \begin{cases} T_{a_1} : \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, a_1) \equiv \mathbf{f}_1, \\ T_{a_2} : \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, a_2) \equiv \mathbf{f}_2, \\ \qquad \vdots \\ T_{a_\tau} : \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, a_\tau) \equiv \mathbf{f}_\tau . \end{cases} \tag{13}$$

Consider the τ functions of the following form:

$$\begin{aligned} \mathbf{F}_1 &= \mathbf{f}_\tau(\mathbf{f}_{\tau-1}(\ldots\mathbf{f}_2(\mathbf{f}_1(\mathbf{x}))\ldots)), \\ \mathbf{F}_2 &= \mathbf{f}_1(\mathbf{f}_\tau(\mathbf{f}_{\tau-1}(\ldots\mathbf{f}_3(\mathbf{f}_2(\mathbf{x}))\ldots))), \\ &\vdots \\ \mathbf{F}_\tau &= \mathbf{f}_{\tau-1}(\mathbf{f}_{\tau-2}(\ldots\mathbf{f}_1(\mathbf{f}_\tau(\mathbf{x}))\ldots)), \end{aligned} \tag{14}$$

where $\mathbf{x} = \{x_1, \ldots , x_n\}$ and $\mathbf{f}_i = \{f_i^{(1)}, \ldots , f_i^{(n)}\}, \mathbf{F}_i = \{F_i^{(1)}, \ldots , F_i^{(n)}\}, \ i = 1, 2, \ldots , \tau$ are the n-component functions. Thus, one can rewrite the perturbed map (12) in the form:

$$\begin{aligned} T_1 &: \mathbf{x} \mapsto \mathbf{F}_1(\mathbf{x}, a_1, \ldots , a_\tau), \\ T_2 &: \mathbf{x} \mapsto \mathbf{F}_2(\mathbf{x}, a_1, \ldots , a_\tau), \\ &\vdots \\ T_\tau &: \mathbf{x} \mapsto \mathbf{F}_\tau(\mathbf{x}, a_1, \ldots , a_\tau), \end{aligned} \tag{15}$$

for which the initial conditions are determined as follows: $\mathbf{x}_1 = \mathbf{f}_1(\mathbf{x}_0)$, $\mathbf{x}_2 = \mathbf{f}_2(\mathbf{x}_1), \ldots, \mathbf{x}_{\tau-1} = \mathbf{f}_{\tau-1}(\mathbf{x}_{\tau-2})$. It turns out that this construction can essentially simplify the study of the perturbed maps. It follows from:

Proposition 2 ([23]). If the map T_k, $1 \le k \le \tau$ has a periodic orbit of the period t and the function $\mathbf{f}_k(\mathbf{x})$ is a C^0-function then the map T_p, $p = k + 1 \pmod{\tau}$, also has a periodic orbit of the same period t. Moreover, if

(i) a periodic orbit of the map T_k is stable then a periodic orbit of the map T_p is stable as well;
(ii) $\mathbf{f}_k$ is a homeomorphism then the maps T_k and T_p are topologically equivalent.

Thus, the initial perturbed map is decomposed into τ independent maps which do not connect with each other, except for the initial conditions. Therefore, instead of the investigation of the perturbed maps (12) or (13) it is sufficient to consider one of the maps $T_1, \ldots, T_\tau$ (15). This fact is the basis of our analytic approach to the problem of the stabilization of chaotic dynamics in certain non-linear systems. It is clear that it is extremely difficult to get analytical results regarding suppression and non-feedback control of the chaotic behaviour in general. However, we can find an exact solution of this problem for some classes of maps. We give elements of such an approach on the examples of one- and two-dimensional maps.

Let us assign the periodic perturbations with period τ to the values $\hat{a} = (a_1, \ldots, a_\tau)$. Then the set $\hat{A} = \{\hat{a} \in \underbrace{A \otimes A \otimes \cdots \otimes A}_{\tau\ times} : a_i \ne a_j,\ 1 \le i, j \le \tau,\ i \ne j\}$ corresponds to a totality of periodic perturbations of the period τ operating on A. Suppose now that every parameter in the set $\{a_i\}$, $i = 1, 2, \ldots, \tau$, corresponds to chaotic dynamics of the unperturbed map (10). In other words, let us introduce into consideration a chaoticity set $A_c \subset A$ such that if in (10) $a = a_i \in A_c$, $1 \le i \le \tau$, then it exhibits chaotic properties and thus, it does not have stable periodic orbits. Furthermore, let us introduce the set $\hat{A}_c = \{\hat{a} \in \underbrace{A_c \otimes A_c \otimes \cdots \otimes A_c}_{\tau\ times} : a_i \ne a_j,\ 1 \le i, j \le \tau,\ i \ne j\}$.

Let us consider the quadratic maps family

$$Q_a : x \mapsto ax(1 - x)\,, \qquad a \in (0, 4], \tag{16}$$

and a family of piecewise linear maps

$$P_a : x \mapsto \begin{cases} q(a)x + r(a)\,, & 0 \le x \le a\,, \\ p(a)(1 - x)\,, & a < x \le 1\,, \end{cases} \tag{17}$$

where $q(a) = (1 - a)/[a(2 - a)]$, $r(a) = 1/(2 - a)$, $p(a) = 1/(1 - a)$. It is known that for map (16) there exists a chaoticity set A_c of the positive measure [24,25]. As to map (17), for any $a \in (0, 1) = A_c$ it has the chaotic behaviour [23].

Theorem 1 ([35,23]). There is a subset $\hat{A}_d$ of the set $\hat{A}_c$ such that if $\hat{a} \in \hat{A}_d$ then the perturbed maps (16) and (17) possess stable periodic orbits.

Illustrations of this theorem for the perturbed map (16) are given in Figs 7 and 8. The analogous statement holds true for a certain class of two-dimensional maps including maps having the most pronounced chaotic behaviour. In turn, exhibition of such strong properties of chaoticity is possible if the system possesses a hyperbolic type of attractor. We shortly clarify the sense of this result for a map with the hyperbolic attractor following the recent results by Loskutov and Rybalko [23].

Remember that a compact invariant set Λ is said to be a hyperbolic attractor if it is an attractor and simultaneously is a hyperbolic set of the dynamical system, i.e. the tangent space is decomposed into two subspaces E^s and E^u which are defined by the facts that infinitesimally close trajectories corresponding to the subspace E^s exponentially converge with $t \to +\infty$, and trajectories corresponding to the subspace E^u exponentially converge with $t \to -\infty$. The hyperbolic attractor is characterized by the property that it is a structural stable subset. Moreover, systems with hyperbolic attractors are models of the physical systems with the rigorous chaotic properties [27,28].

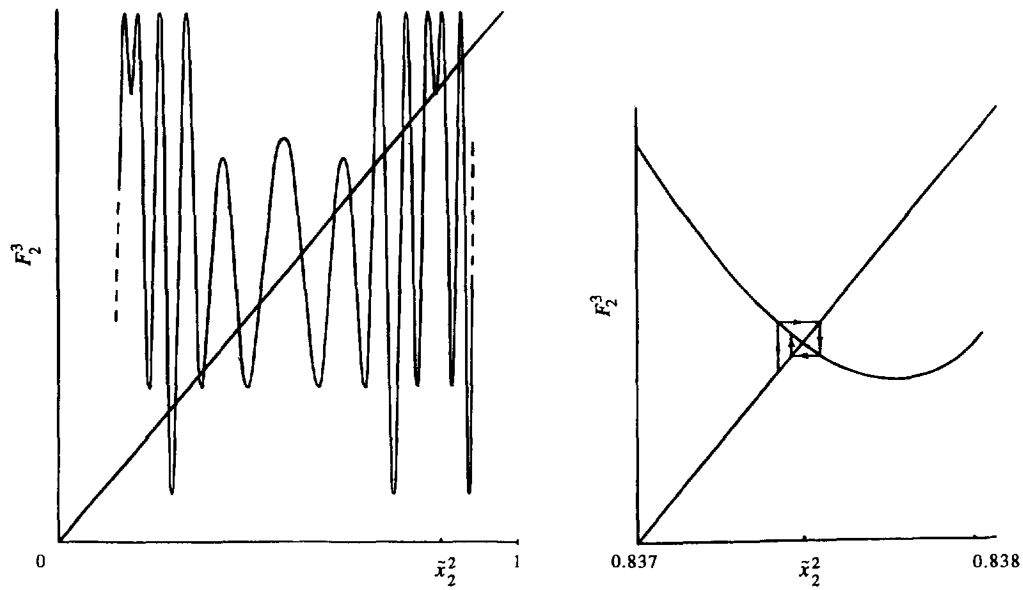

Fig. 7. One of the stable fixed points of two-periodic perturbed map (16), $a_1 = 3.67857336\ldots$, $a_2 = 3.97459125\ldots$, $a_1, a_2 \in A_c$.

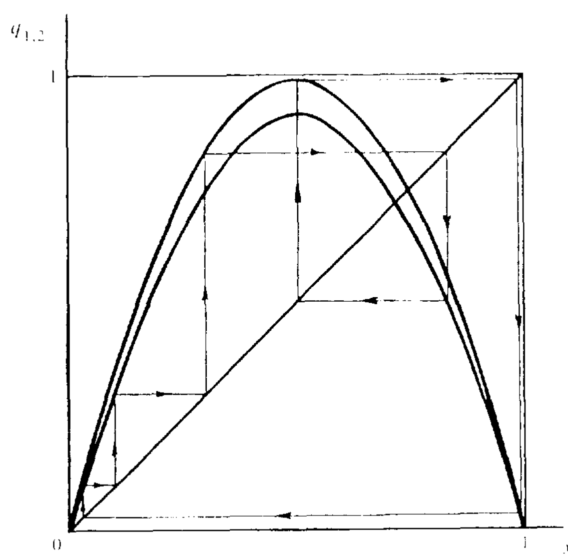

Fig. 8. The stable periodic orbit of two-periodic perturbed map (16).

Consider a quite simple construction of the map having the hyperbolic attractor [29] which has been proposed by [30] in the context of investigation of radio physical systems.

Let $Q = \{(x, y) : |x| \le 1, \ |y| \le 1\}$ be a square in the plane (x, y). Then the hyperbolic map T is given by the following construction:

$$T:\ (x,y)\mapsto f(x,y)=\begin{cases}(\lambda_1(x+1)-1,\ (y+1)/\lambda_2-1)\,, & (x,y)\in Q_1,\\ (\lambda_3(x-1)+1,\ (y-1)/\lambda_4+1)\,, & (x,y)\in Q_2,\end{cases} \tag{18}$$

and the areas Q_1, Q_2 are presented as the separation of the square Q by the function $u(x)$ into two parts,

$$Q_1=\{(x,y)\in Q:\ y<u(x)\}\,,$$
$$Q_2=\{(x,y)\in Q: y>u(x)\}\,,$$

where $u(x)$ is defined in the interval $[-1,1]$ such that $|\,u(x)\,|<1$. In addition, the constants $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ and the function $u(x)$ are chosen in such a way that $TQ_1\subset Q$, $TQ_2\subset Q$.

We consider the simplest case $u(x)=ax$, $\lambda_1=\lambda_3$, $1/\lambda_2=1/\lambda_4\equiv\lambda_2$, i.e.

$$T:\ (x,y)\mapsto f(x,y)=\begin{cases}(\lambda_1(x+1)-1,\ \lambda_2(y+1)-1)\,, & y<ax,\\ \\ (\lambda_1(x-1)+1,\ \lambda_2(y-1)+1)\,, & y>ax,\end{cases} \tag{19}$$

$|\,a\,|<1$. It is not difficult to obtain the existence conditions for a hyperbolic attractor in map (19). It is the fulfillment of the following inequalities:

$$\begin{gathered}0<\lambda_1<1/2\,,\\ 1<\lambda_2<2/(1+|\,a\,|)\,,\\ |\,a\,|<1\,.\end{gathered} \tag{20}$$

Now it is easy to generalize map (19) for the case $|\,a\,|>1$. It can be done by the substitution $x\to y$ and $a\to 1/a$. Thus,

$$T:\ (x,y)\mapsto f(x,y)=\begin{cases}(\lambda_1(x+1)-1,\ \lambda_2(y+1)-1), & y>ax,\\ \\ (\lambda_1(x-1)+1,\ \lambda_2(y-1)+1), & y<ax.\end{cases} \tag{21}$$

In this case the hyperbolicity condition (20) is rewritten as follows:

$$\begin{gathered}1<\lambda_1<2/(1+1/\,|\,a\,|\,,\\ 0<\lambda_2<1/2\,,\\ |\,a\,|>1\,.\end{gathered} \tag{22}$$

We study the simplest case of two-periodic perturbation, $\hat{a}=(a_1,a_2)\in\hat{A}_c$. Then the perturbed map (19), (21) is constructed as follows. First of all, it is necessary to take into account that in switching the control parameter we must comply with the condition of hyperbolicity. It is possible if the parameter a is altered near the value $a=1$, i.e. $a_1<1$ and $a_2>1$. Simultaneously it is necessary to vary the parameters λ_1 and λ_2. Observing these conditions, the perturbed map is written as follows:

$$\bar{T}\ :\ \begin{cases}(x,y)\mapsto f(a_2,\lambda_1^2,\lambda_2^2,x,y)\circ f(a_1,\lambda_1^1,\lambda_2^1,x,y)\\ (x,y)\mapsto f(a_1,\lambda_1^1,\lambda_2^1,x,y)\circ f(a_2,\lambda_1^2,\lambda_2^2,x,y)\end{cases}. \tag{23}$$

One can see that the points $P_1=(1,1)$ and $P_2=(-1,-1)$ are the fixed ones for both the unperturbed map under the conditions (20), (22) and the perturbed map. It is obvious that for the unperturbed hyperbolic map they are unstable. Let us show that in map (23) P_1, P_2 are *stable* fixed points. Really, the differential of the perturbed map (for the add and the even iterations) is the following matrix:

$$D\bar{T}=\begin{pmatrix}\lambda_1^2 & 0\\ 0 & \lambda_2^2\end{pmatrix}\cdot\begin{pmatrix}\lambda_1^1 & 0\\ 0 & \lambda_2^1\end{pmatrix}=\begin{pmatrix}\lambda_1^2\lambda_1^1 & 0\\ 0 & \lambda_2^2\lambda_2^1\end{pmatrix}\equiv\begin{pmatrix}\lambda_1^* & 0\\ 0 & \lambda_2^*\end{pmatrix}. \tag{24}$$

Therefore its eigenvalues λ_1^* and λ_2^* (see (20), (22)) are varied in the range $0 < \lambda_1^* < 1/(1 + 1/|a_2|)$, $0 < \lambda_2^* < 1/(1+|a_1|)$. It means that $|\lambda_1^*| < 1$, $|\lambda_2^*| < 1$, i.e. the fixed points P_1 and P_2 of the perturbed hyperbolic map become stable after perturbation, and almost all trajectories are attracted to them with time.

Therefore, in spite of the fact that the system dynamics has the most pronounced chaotic properties, there are parametric perturbations capable to stabilize such a type of the behaviour. However, as to stabilization of unstable periodic orbits embedded into hyperbolic attractor up to the present this problem remains to be explored in the framework of the described approach.

4. CONCLUDING REMARKS

Thus, we have shown that for a class of dynamical systems it is possible to find multiplicative perturbations leading to stabilization of initially unstable behaviour. The described approach has an advantage over the other ways because it gives us a method of the analytical detection of the possibility of chaos suppression for a practically arbitrary map under external perturbations. In fact, introducing τ functions of the form (14) one can rewrite the perturbed map as (15). Beginning with the first iteration $t = 1$ and step-by-step searching through the parameters $a_1, ..., a_\tau$ in the set A_c, all fixed points of the map T_k^t should be determined and among them the stable ones should be found. If at the given t the map T_k^t does not have the stable fixed points then it is necessary to increase t by the unit and to repeat the whole calculation. This procedure is followed until the values $a_1, ..., a_\tau \in A_c$ are found.

Besides, by means of the solution of the inverse problem for the perturbed dynamical system decomposed into τ subsystems, the present feedback-free method allows us to stabilize the desired periodic orbits, i.e. to realize the goal oriented control. Thus, we can apply the described theoretical approach to the development of the non-feedback controlling behaviour of dynamical systems [31]. Moreover, the obtained results can be applied to the problem of information processing [26], chains of coupled automata and the self-organization problem [33,21,34].

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REFERENCES

1. T. Shinbrot, *Nonlinear Sci. Today* **3,** 1–8 (1993).
2. T. Shinbrot, C. Grebogi, E. Ott and J. A. Yorke, *Nature* **363,** 411–417 (1993).
3. S. Rajasekar and M. Lakshmanan, *Physica D* **67,** 282–300 (1993).
4. E. Ott, C. Grebogi and J. Yorke, *Phys. Rev. Lett.* **64,** 1196–1199 (1990).
5. U. Dressler and G. Nitshe, *Phys. Rev. Lett.* **68,** 1–4 (1992).
6. I. M. Starobinets and A. S. Pikovsky, *Phys. Lett. A* **181,** 149–152 (1993).
7. V. V. Alekseev and A. Yu. Loskutov, *Moscow Univ. Phys. Bull.* **40,** 46–49 (1985).
8. V. V. Alekseev and A. Yu. Loskutov, *Sov. Phys.-Dokl.* **32,** 270–271 (1987).
9. R. Lima and M. Pettini, *Phys. Rev. A* **41,** 726–733 (1990).
10. Yu. S. Kivshar, F. Rödelsperger and H. Benner, *Phys. Rev. E* **49,** 319–324 (1994).
11. A. Hübler, *Helv. Phys. Acta* **62,** 343–346 (1989).
12. E. Lüsher and A. Hübler, *Helv. Phys. Acta* **62,** 544–551 (1989).
13. E. A. Jackson and A. Hübler, *Physica D* **44,** 407–420 (1990).
14. E. A. Jackson and A. Hüble, *Physica D* **54,** 253–265 (1992).
15. K. Pyragas, *Phys. Lett A* **170,** 421–424 (1992).
16. K. Pyragas, *Z. Naturforsh A* **48,** 629–632 (1993).
17. Yu. I. Neymark and P. S. Landa, *Stochastic and Chaotic Oscillations.* Kluwer, Amsterdam (1992).
18. N. N. Bogoliubov and Yu. A. Mitropolskii, *Asymptotical Methods in the Theory of Nonlinear Oscillations.* Nauka, Moscow (1974).
19. O. E. Rössler, *Phys. Lett. A* **57,** 397–398 (1976).
20. D. Gurel and O. Gurel, *Oscillations in Chemical Reactions.* Springer, Berlin (1983).

21. A. Yu. Loskutov, In preparation.
22. A. Yu. Loskutov, *J. Phys. A* **26,** 4581–4594 (1993).
23. A. Yu. Loskutov and S. D. Rybalko, Preprint ICTP, No. IC/94/347. Trieste, Italy (November 1994).
24. M. Jakobson, *Commun. Math. Phys.* **81,** 39–88 (1981).
25. W. de Melo and S. van Strien, *One-dimensional Dynamics*. Springer, Berlin (1993).
26. A. Yu. Loskutov and V. M. Tereshko, *Proc. of the SPIE Annual Meeting*, San Diego, CA, 11–16 July 1993, Vol. 2038, pp. 263–272 (1993).
27. V. S. Afraimovich, In *Nonlinear Waves 1. Dynamics and Evolution*, edited by J. Engelbreht, A. V. Gaponov-Grekhov and M. I. Rabinovich, pp.14–28. Springer, Berlin (1989).
28. Ya. G. Sinai (Ed.), *Dynamical Systems II*. Springer, Berlin (1989).
29. E. A. Sataev, *Uspekhi Mat. Nauk* **47,** 147–202 (1992) (in Russian).
30. V. P. Belykh, *Radio and Svjaz'*, pp. 161–176. Nauka, Moscow (1982) (in Russian).
31. A. Yu. Loskutov and V. M. Tereshko, In preparation.
32. A. Yu. Loskutov and V. M. Tereshko, In preparation.
33. A. Yu. Loskutov and G. E. Thomas, *Proc. of the SPIE Annual Meeting*, San Diego, CA, 11–16 July 1993, Vol. 2037, pp. 238–249 (1993).
34. A. Yu. Loskutov, V. M. Tereshko and K. A. Vasiliev, *Int. J. Neural Systems*. **6,** 175–186 (1995).
35. A. Yu. Loskutov a,d A. I. Shishmarev, *Uspekhi Mat. Nauk* **48,** 169–170 (1993).