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PARAMETRIC PERTURBATIONS AND NON–FEEDBACK CONTROLLING CHAOTIC MOTION

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Abstract. In this paper we generalize analytic studies the problems related to suppression of chaos and non–feedback controlling chaotic motion. We develop an analytic method of the investigation of qualitative changes in chaotic dynamical systems under certain external periodic perturbations. It is proven that for polymodal maps one can stabilize chosen in advance periodic orbits. As an example, the quadratic family of maps is considered.

Also we demonstrate that for a piecewise linear family of maps and for a two-dimensional map having a hyperbolic attractor there are feedback-free perturbations which lead to the suppression of chaos and stabilization of certain periodic orbits.

1. **Introduction.** In the last few years an unexpectedly interesting property of chaotic dynamical systems has come to light: weak external perturbations can qualitative change their behavior. It was found that chaotic dynamics is surprisingly pliable to certain actions. Such problems as controlling chaotic motion [1] and suppression of chaos [3, 4, 2] are related to this phenomenon. Due to important and profitable applications (see [5] and refs. cited therein) they have attracted a considerable interest. It is sufficient to mention the following: information processing and communications (see, e.g., [6, 7, 8, 10]), an artificial pattern formation in chaotic spatio–temporal systems [9], a self–organization problem [11, 12], controlling cardiac chaos and suppression of complex cardiac rhythms (see, e.g., [13, 14, 15, 16, 17, 18, 19] and also references therein), and other significant problems of nonlinear dynamics [5].

Apparently, the control of dynamical systems by parametric perturbations has been first proposed in papers [3, 4]. Later, in a series of publications (see [2] and refs. cited therein) this phenomenon has been justified analytically and studied numerically.

As is known, qualitative changes in the behavior of chaotic systems with external perturbations can be realized by two different ways. First of them provides the removing a system from the chaotic state into a regular regime by means of external actions without a feedback. In other words, this method does not take into account the current value of the dynamical variables of the system. The qualitative different method ensures the correction in dynamical variables according to their values and

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thus, this method involves the feedback as a component of the system. By the established convention, the first way is called suppression of chaos (or sometimes non–feedback controlling chaotic motion) and the second one is called (feedback) controlling chaos. In turn, each of these ways can be subdivided into a parametric (multiplicative) method and a forcing method.

Although there are several papers devoted to a theoretical substantiation of the questions of the control (as reviews see [5, 2]), it would be interesting to develop a sequential theory and establish a rigorous foundation of the possibility of the chaos suppression and its control. Apparently, it is very difficult to make this in general. But we may resolve such a problem for certain families of dynamical systems. These questions are a major focus of interests for the present paper. We describe rigorous methods of the investigation of qualitative changes in the behavior of chaotic maps under external periodic perturbations and propose an analytical key which allows to find such perturbations.

The article is organized as follows. First, n-dimensional maps in a general form with external τ–periodic parametric perturbations are considered. Then it is shown that for polymodal maps there are τ-periodic parametric perturbations which lead to stabilization of *prescribed* periodic orbits. As an example, a quadratic family of maps is considered in detail. In the next sections a one-dimensional piecewise–linear family and a two–dimensional map having a hyperbolic type of attractor are studied. It is shown that chaotic behavior of these maps may be suppressed by the stabilization of certain periodic orbits.

2. **External perturbations of n-dimensional maps.** In general, the control (feedback or feedback–free) of dynamical systems involves a certain additive or/and multiplicative variable(s) which take(s) into account additive or/and multiplicative perturbations, respectively. Therefore, first we study the properties of dynamical systems with external perturbations.

Let us consider a n-dimensional one–parametric family of maps in a general form

$$T_a: \ \mathbf{x} \longmapsto \mathbf{f}(\mathbf{x}, a) \ , \tag{1}$$

where $\mathbf{x} = \{x_1, \ldots, x_n\} \in M$, $\mathbf{f} = \{f_1, \ldots, f_n\}$ is a certain nonlinear function, a is a control parameter, and M is a compact invariant set. Let A be a set of the admissible values of the parameter a. Introduce a parametric perturbation for the map (1), $G : A \to A$,

$$G: \ a \longmapsto g(a), \qquad a \in A. \tag{2}$$

In this case the perturbed map (1) has the form

$$\mathbf{T}: \ \mathbf{y} \longmapsto \mathbf{h}(\mathbf{y}) \ , \tag{3}$$

where $\mathbf{y} \in M \times A$, $\mathbf{y} = (\mathbf{x}, a)$, $\mathbf{h}(\mathbf{y}) = (\mathbf{f}(\mathbf{x}, a), g(a))$. We may involve parametric perturbations into the system by two different ways. If an external source is associated with a multiplicative action with respect to dynamical variables then (multiplicative) parameters of the system are modified. If, however, external sources are included into the model as additive terms with respect to dynamical variables then the force perturbation takes place. Thus, depending on the type of external actions, a certain component of the perturbed dynamical systems is modified.

Below only periodic (cyclic) perturbations are considered. Then for period τ we have from (2): $a_{i+1} = g(a_i)$, $i = 1, 2, \ldots, \tau - 1$, $a_1 = g(a_\tau)$, $a_i \neq a_j$ for $i \neq j$, $a_i \in A$, $1 \leq i \leq \tau$. Now, any type of periodic perturbations with period

τ can be associated with the following vectors: $\hat{a} = (a_1, \ldots, a_\tau),\ \hat{a} \in \mathbb{R}^\tau$. So, the set $\hat{A} = \{\hat{a} \in \underbrace{A \otimes A \otimes \cdots \otimes A}_{\tau \text{ times}} :\ \hat{a} = (a_1, \ldots, a_\tau),\ a_i \neq a_j,\ 1 \leq i, j \leq \tau,\ i \neq j,$ $a_1, \ldots, a_\tau \in A\}$, $\hat{A} \subset \mathbb{R}^\tau$, corresponds to all possible sets of τ-periodic perturbations which operate on A. Note that the explicit form of the function $g(a)$ in (2) is not essential. For our aims it is sufficient to know that g transforms the parameter values into an infinite series $\{a_i\}$ consisting of the repeated subsequences $(a_1, a_2, \ldots, a_\tau)$.

It is quite clear that period t of any obtained periodic orbit in the perturbed map (3) is multiple to the period τ of perturbation, $t = \tau k$, where k is a positive integer. Really, if the perturbed map (3) has a t-periodic orbit then the coordinate sequences which form this periodic orbit are also periodic with period t. But the sequence $\{a_i\}$ is already periodic with period τ. Therefore, $t = \tau k$.

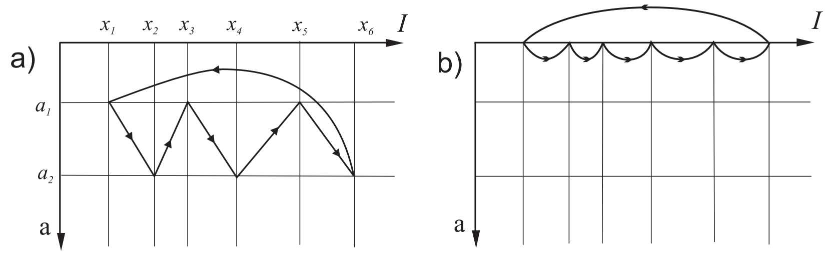

FIGURE 1. A periodic orbit of the periodically perturbed map (3) in the space (x, a) (a), and in the projection onto the coordinate axis (b).

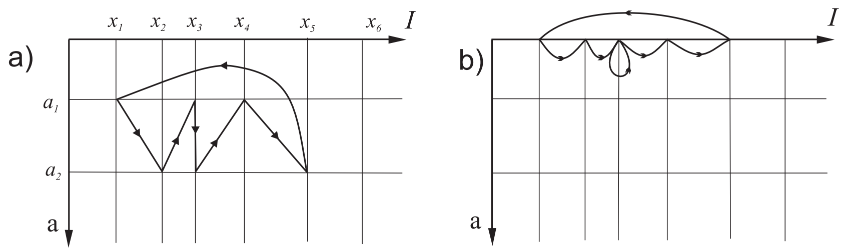

FIGURE 2. A periodic orbit with coincident coordinates of the periodically perturbed map (3) in the space (x, a) (a), and in the projection onto the coordinate axis (b).

However, there is an important remark to be made. If we project an obtained periodic orbit onto the initial space M (i.e. when the perturbed system (3) as a non–autonomous one is considered) then we may get in M an orbit which can not be a periodic orbit in the usual sense. The matter is that the points of this periodic orbit, which differ from each other only in the coordinate a, are projected onto one and the same point in the space M. Thus, in non-autonomous systems such a behavior can not correspond to periodic orbits because the representative point of the map hits several times in some points forming the given periodic orbit.

For example, for one–dimensional ($n = 1$) maps, in a general case, in the projection onto the initial space $M = I$ we get a periodic orbit of the period τk (Fig.1a,b). But in I we may obtain a orbit with the coincided x–coordinates when $x_i = x_m,\ a_i \neq a_m,\ i \neq m$, where (x_i, a_i) and (x_m, a_m) are points of the periodic orbit of the perturbed map (3) (Fig.2a,b). However, the described situation can be considered as a degenerated one, and it may occur only in special cases.

Introduce a quite simple construction which is necessary for the further analysis. Any τ-cyclic transformation for the map (1) means that the perturbed system (3) can be written as follows:

$$\mathbf{T} \;=\; \begin{cases} T_{a_1} : \mathbf{x} \longmapsto \mathbf{f}(\mathbf{x}, a_1) \equiv \mathbf{f}_1 \ , \\ T_{a_2} : \mathbf{x} \longmapsto \mathbf{f}(\mathbf{x}, a_2) \equiv \mathbf{f}_2 \ , \\ \dots\dots\dots\dots\dots\dots \ , \\ T_{a_\tau} : \mathbf{x} \longmapsto \mathbf{f}(\mathbf{x}, a_\tau) \equiv \mathbf{f}_\tau \ . \end{cases} \tag{4}$$

Consider τ functions of the following form:

$$\begin{aligned} &\mathbf{F}_1 = \mathbf{f}_\tau(\mathbf{f}_{\tau-1}(...\mathbf{f}_2(\mathbf{f}_1(\mathbf{x}))...)) \ , \\ &\mathbf{F}_2 = \mathbf{f}_1(\mathbf{f}_\tau(\mathbf{f}_{\tau-1}(...\mathbf{f}_3(\mathbf{f}_2(\mathbf{x}))...))) \ , \\ &\dots\dots\dots\dots\dots\dots\dots \ , \\ &\mathbf{F}_\tau = \mathbf{f}_{\tau-1}(\mathbf{f}_{\tau-2}(...\mathbf{f}_1(\mathbf{f}_\tau(\mathbf{x}))...) \ , \end{aligned} \tag{5}$$

where $\mathbf{x} = \{x_1, \dots, x_n\}$, and $\mathbf{f}_i = \{f_i^{(1)}, \dots,\ f_i^{(n)}\},\ \mathbf{F}_i = \{F_i^{(1)}, \dots, F_i^{(n)}\},\ i = 1, 2, \dots, \tau$, are the n-component functions. Thus, the perturbed map (3) one can rewrite in the form

$$\begin{aligned} &T_1 : \ \mathbf{x} \longmapsto \mathbf{F}_1(\mathbf{x}, a_1, \dots, a_\tau) \ , \\ &T_2 : \ \mathbf{x} \longmapsto \mathbf{F}_2(\mathbf{x}, a_1, \dots, a_\tau) \ , \\ &\dots\dots\dots\dots\dots\dots\dots \ , \\ &T_\tau : \ \mathbf{x} \longmapsto \mathbf{F}_\tau(\mathbf{x}, a_1, \dots, a_\tau) \ , \end{aligned} \tag{6}$$

for which initial conditions are determined as follows: $\mathbf{x}_1 = \mathbf{f}_1(\mathbf{x}_0),\ \mathbf{x}_2 = \mathbf{f}_2(\mathbf{x}_1), \dots,$ $\mathbf{x}_{\tau-1} = \mathbf{f}_{\tau-1}(\mathbf{x}_{\tau-2})$.

Thus, analysis of non-autonomous τ-periodic perturbed maps may be reduced to the consideration of τ autonomous maps of the form (6). Moreover, to describe the dynamics of the initial non–autonomous system it is sufficient to carry out the analysis of only one of the functions (5) [20]. This result follows from

Proposition 1. *If the map $T_k,\ 1 \leq k \leq \tau$ has a periodic orbit of period t and the function $\mathbf{f}_k(\mathbf{x})$ is a C^0-function then the map $T_p,\ p = k + 1 \pmod{\tau}$, also has a periodic orbit of the same period t. Moreover, if*

i) a periodic orbit of the map T_k is stable then a periodic orbit of the map T_p is stable as well;

ii) $\mathbf{f}_k$ is a homeomorphism then the maps T_k and T_p are topologically equivalent.

Proof see in [20].

3. **Stabilization of the required periodic orbits and suppression of chaos.** In this Sect. we demonstrate that for polymodal maps one can stabilize the prescribed periodic orbits by means of simple periodic parametric perturbations. It is also shown that for a certain class of the piecewise linear maps and for a so-called Belykh map, which has a hyperbolic attractor, such perturbations lead to the chaos suppression and stabilization of certain initially unstable orbits.

Since we consider chaotic maps, let us introduce a subset $A_c \subset A$ such that if $a \in A_c \subset A$ then the map (1) possesses chaotic dynamics. A rigorous definition of chaos may be found in [2, 23, 24].

3.1. **Polymodal maps.** Consider a one–dimensional map in a general form

$$T_a : x \longmapsto f(x, a), \tag{7}$$

where $x \in I$, f is a certain function and a is a control parameter. As before, let us introduce a τ-periodic parametric perturbation, $G : A \to A$, $G :\ a \longmapsto g(a),\ a \in A \subseteq A$, such that $a_{i+1} = g(a_i)$, $i = 1, 2, \ldots, \tau - 1$, $a_1 = g(a_\tau)$, $a_i \neq a_j$ for $i \neq j$. Rewrite the perturbed map in the form:

$$\mathbf{T} = \begin{cases} x \longmapsto f(a, x)\ , \\ \\ a \longmapsto g(a)\ . \end{cases} \tag{8}$$

Now we can get the following interesting result. Suppose that the map (8) satisfies the following conditions:

(i) there is a subset $\sigma \subset I$ such that for any $x_1, x_2 \in \sigma$ we have $a^* \in A$ for which $f(x_1, a^*) = x_2$;

(ii) for an arbitrary $a \in A$ there is a critical point $x_c \in \sigma$, i.e. $\partial f(x, a)/\partial x|_{x=x_c} \equiv \mathrm{D}_x f(x_c, a) = 0$.

It is quite clear that in this case for any $x_2, x_3, \ldots, x_\tau \in \sigma$ we can find such values x_1 and $a_1, a_2, \ldots, a_\tau$ that in the perturbed map $\mathbf{T}$ the periodic orbit $(x_1, x_2, \ldots, x_\tau)$ is stable for $\hat{a} = (a_1, \ldots, a_\tau)$. Really, let us choose the following arbitrary values $x_1, x_2, \ldots, x_\tau$. In view of the condition (i), the system

$$f(x_1, a_1) = x_2,\ f(x_2, a_2) = x_3,\ \ldots,\ f(x_\tau, a_\tau) = x_1 \tag{9}$$

has the solution of the form $\hat{a} = (a_1, a_2, \ldots, a_\tau)$. This means that the sequence of the points $(x_1, x_2, \ldots, x_\tau) = p$ is a τ-periodic orbit of the map $\mathbf{T}$ at the periodic perturbation $\hat{a} = (a_1, a_2, \ldots, a_\tau)$. To stabilize this orbit p it is sufficient to choose the element x_1 in a quite small neighborhood of the critical value x_c because the corresponding multiplier $\beta(p) = \prod_{i=1}^{N} \mathrm{D}_x f(x_i, a_i)$ and $\mathrm{D}_x f(x_c, a) = 0$ for an arbitrary a. Thus, the orbit $(x_c, x_2, \ldots, x_\tau) = p$ is stable for $x_i \in \sigma \subset M$, $i = 1, 2, \ldots, \tau$, and parameters $\hat{a} = (a_1, a_2, \ldots, a_\tau)$ satisfying the system (9).

3.1.1. *Example.* Consider as an example the following quadratic family of maps in the form $T_a : [0, 1] \to [0, 1]$,

$$T_a :\ x\ \longmapsto\ f(a, x)\ =\ ax(1 - x)\ , \tag{10}$$

that is a simplest model of some nonlinear phenomena (see, e.g., [21, 22, 23, 24]). It is well known that for $a \in [0, a_\infty)$, $a_\infty = 3.569\ldots$, the map (10) has only regular behavior. However for $a \in (a_\infty, 4]$ it can be both regular and chaotic. Let us introduce a subset $A_c \subset (a_\infty, 4]$ such that for $a \in A_c$ the map (10) has the chaotic behavior. It can be proven [25, 26] that the subset A_c has the positive Lebesgue measure and the point $a = 4$ is a density point of this subset.

Let us consider the periodically perturbed map (10). In previous papers (see [27, 2] and references cited therein) it has been shown that periodic parametric perturbations operating in the chaotic subset $\hat{A}_c$ can stabilize its dynamics.

Theorem 1. *There exists a subset $\hat{A}_d \subset \hat{A}_c$ such that if $\hat{a} \in \hat{A}_d$ then the perturbed quadratic family possesses stable periodic orbits.*

Proof see in [27]. It is evident that for the quadratic family (10) the set σ is the interval $[x_b, x_e]$, where x_b and x_e are the solutions of the equation $x_{int} = f(x, 4)$, and x_{int} is the intersection point of the curves $y = 4x(1-x)$ and $y = x$. Thus, $[x_b, x_e] = [1/4, 3/4]$. Suppose that this map has an orbit of the period $t = \tau$, i.e. $p = (x_1, x_2, \ldots, x_t)$. Then the points forming this periodic orbit obey the following system:

$$x_2 = a_1 x_1(1-x_1), \quad x_3 = a_2 x_2(1-x_2), \quad \ldots, \quad x_1 = a_t x_t(1-x_t). \tag{11}$$

To determine the parameters for which the perturbed map has the periodic orbit $p = (x_1, x_2, \ldots, x_t)$, it is necessary to express the values a_i from system (11):

$$a_1 = \frac{x_2}{x_1(1-x_1)}, \quad a_2 = \frac{x_3}{x_2(1-x_2)}, \quad \ldots, \quad a_t = \frac{x_1}{x_t(1-x_t)}. \tag{12}$$

It is obvious that not for all possible $x_i \in (0,1)$ the expression $a_i \in [0,4]$ takes place. However, if it holds then for a given periodic orbit $p = (x_1, x_2, \ldots, x_t)$ one can find the parameter values $(a_1, a_2, \ldots, a_t)$ for which the perturbed map has such an orbit. If $|\beta(p)| = \left|\prod_{i=1}^{t} a_i(1-2x_i)\right| < 1$ then it is stable. Therefore, from (12) we get:

$$|\beta(p)| = \left|\prod_{i=1}^{t} \frac{1-2x_i}{1-x_i}\right| < 1 \ . \tag{13}$$

Owing to the equality $x_c = 1/2$ we find that $(1-2x_c)/(1-x_c) = 0$. Thus, the expression (13) always holds.

The set of values $(x_1, x_2, \ldots, x_t)$, for which there are $a_i \in [0,4]$ and inequality (13) holds, forms a certain region in the coordinate space $\mathbb{R}^t$. Using the system (12) we can obtain the corresponding region in the parametric space $\mathbb{R}^t$. Let us consider the case of two–periodic perturbation, $\tau = 2$. In the space (x_1, x_2) the region of stable orbits is defined by the following set of inequalities:

$$0 < \frac{x_2}{x_1(1-x_1)} \le 4, \quad 0 < \frac{x_1}{x_2(1-x_2)} \le 4, \quad \left|\frac{1-2x_1}{1-x_1}\frac{1-2x_2}{1-x_2}\right| < 1 \ . \tag{14}$$

Realization of the first and the second inequalities corresponds to the region of all orbits of period two. The third inequality cuts from it the existence region of the *stable* orbits.

Let us fix the value $x_1 \in (0,1)$. Taking into account singularities, we find from (14):

$$\begin{aligned} 0 < x_2 < \frac{3x_1-2}{5x_1-3} \ , &\quad 0 < x_1 < \frac{1}{3} \ , \\ 0 < x_2 < \frac{x_1}{3x_1-1} \ , &\quad \frac{1}{3} < x_1 < \frac{3}{5} \ , \\ \frac{3x_1-2}{5x_1-3} < x_2 < \frac{x_1}{3x_1-1} \ , &\quad \frac{3}{5} < x_1 < 1 \ . \end{aligned}$$

Thus, we get the existence region of all possible stable orbits of period two, $p = (x_1, x_2)$, for the perturbed map (Fig.3a). To construct a parametric region corresponding to these orbits, it is necessary to transform Fig.3a by means of (12). Let us divide the region in Fig.3a into subregions which are mapped onto (a_1, a_2) by a one-to-one manner. Now it is necessary to transform the boundaries of the subregions. Thus, we find the existence region of the stable period-two orbits, $p = (x_1, x_2)$, in the parametric space (a_1, a_2) (Fig.3b). To clarify the action of the map (11), the region in Fig.3a is divided into 4 parts which marked by different shading. The corresponding regions in Fig.3b have the same structure. Now we can easy analyze the two–periodically perturbed quadratic family.

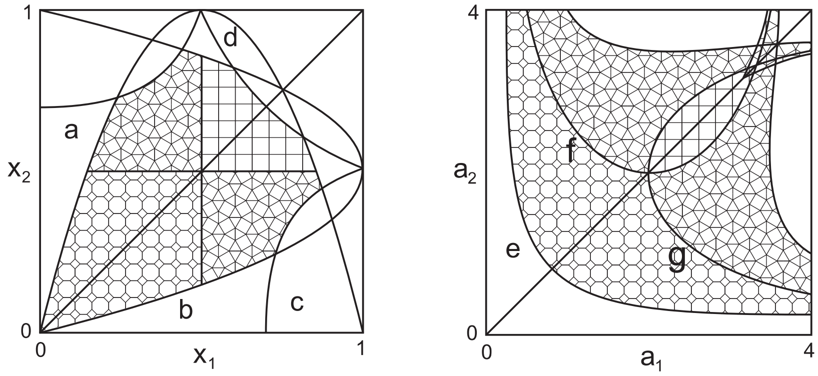

FIGURE 3. The existence region of the period two orbits for the perturbed ($\tau = 2$) quadratic map in the space (x_1, x_2) defined by the curves $x_2 = 4x_1(1 - x_1)$ (a), $x_1 = 4x_2(1 - x_2)$ (b), $x_2 = (3x_1 - 2)/(5x_1 - 3)$ (c), $x_2 = x_1/(3x_1 - 1)$ (d), and in the parametric space (a_1, a_2) given by the curves $a_2 = 1/a_1$ (e), $a_2 = 8/[a_1(4 - a_1)]$ (f), $a_1 = 8/[a_2(4 - a_2)]$ (g).

1) Obviously, the symmetry of the parameter space is the reason of the symmetry of the two–dimensional stability diagrams.

2) Because Fig.3b has intersecting subregions, then the map (11) is a single-valued map but it is not a one-to-one transformation.

3) Intersections mean that the perturbed map is bi-stable. At certain parameter values it may simultaneously have two stable periodic orbits.

3.1.2. *Estimation of admissible noises.* Numerical approach allows us to make only approximate calculations. Therefore, it is necessary to know estimation of the admissible errors in the parameter values of the perturbed map. This estimation can be done using the following result.

Theorem 2. [20] *Suppose that* $f(x, a) \in C^2[M \times A]$ *and at* $\hat{a} = (a_1, a_2, \ldots, a_t)$ *the perturbed map* $\mathbf{T}$ *has a stable orbit of the period* t, $p = (x_1, x_2, \ldots, x_t)$. *Then , if* $|\Delta a_i| \leq \delta_a = 1 \Big/ \left(t S_a L S_x^{t-1} \sum_{i=1}^{t} S_x^i \right)$, *where* $i = 1, 2, \ldots, t$, $S_a = \max_{x,a} |\mathrm{D}_a f(x, a)|$, $L = \max_{x,a} |\mathrm{D}_x^2 f(x, a)|$, $S_x = \max_{x,a} |\mathrm{D}_x f(x, a)|$, *this map has also a stable orbit* $p' = (x_c + \Delta x_1, x_2 + \Delta x_2, \ldots, x_t + \Delta x_t)$ *of period* t *at* $\hat{a}' = (a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_t +$

Δa_t) *with*

$$|\Delta x_i| \le \delta_x = \frac{1}{LS_x^{t-1}} \ .$$

Proof. Assume that all parameters a_i are perturbed, $a_i' = a_i + \Delta a_i$. Let us find a value $\Delta x_1 = x_1' - x_c$, where $x_1' = F_1(x_1', a_1', a_2', \dots, a_t')$. Then $x_c + \Delta x_1 \simeq F_1(x_c, a_1, a_2 \dots, a_t) + \mathrm{D}_x F_1(x_c, \hat{a})\Delta x_1 + \sum\limits_{i=1}^{t} \mathrm{D}_{a_i} F_1(x_c, \hat{a})\Delta a_i$. Therefore, taking into account the expressions $x_c = F_1(x_c, \hat{a})$ and $\mathrm{D}_x F_1(x_c, \hat{a}) = \beta(p) = 0$ we get: $\Delta x_1 = \sum\limits_{i=1}^{t} \prod\limits_{l=i+1}^{t} \mathrm{D}_x f(x_l, a_l) \mathrm{D}_a f(x_i, a_i) \Delta a_i$. Thus,

$$|\Delta x_1| \le \delta_a \sum_{i=1}^{t} \prod_{l=i+1}^{t} \Big|\mathrm{D}_x f(x_l, a_l\Big| \cdot \Big|\mathrm{D}_a f(x_i, a_i)\Big| \le \delta_a t S_a \sum_{i=1}^{t} S_x^i \ . \tag{15}$$

Let us estimate the change in the multiplier of the orbit. We have: $\beta(p') = \sum\limits_{i=1}^{t} \mathrm{D}_x^2 f(x_i, a_i) \prod\limits_{\substack{l=1 \\ l \ne i}}^{t} \mathrm{D}_x f(x_l, a_l)\Delta x_i + \sum\limits_{i=1}^{t} \mathrm{D}_{ax}^2 f(x_i, a_i) \prod\limits_{\substack{l=1 \\ l \ne i}}^{t} \mathrm{D}_x f(x_l, a_l)\Delta a_i$. Obviously, in both of these sums only the first components are nonzero because $\mathrm{D}_x f(x_1, a_1) = \mathrm{D}_x f(x_c, a_1) = 0$. Thus, $\beta(p') = \Big[\mathrm{D}_x^2 f(x_c, a_1)\Delta x_1 + \mathrm{D}_{ax}^2 f(x_c, a_1)\Delta a_1\Big] \prod\limits_{l=2}^{t} \mathrm{D}_x f(x_l, a_l)$. But $\mathrm{D}_{ax}^2 f(x_c, a_1) = \mathrm{D}_a\Big(\mathrm{D}_x f(x_c, a)\Big)\Big|_{a=a_1} = \mathrm{D}_a(0) = 0$. Therefore, the multiplier of the changed periodic orbit is $|\beta(p')| = |\Delta x_1| \Big|\mathrm{D}_x^2 f(x_c, a_1)\Big| \prod\limits_{l=2}^{t} \Big|\mathrm{D}_x f(x_l, a_l)\Big|$. For the stability of this orbit it is necessary that $|\Delta x_1| \Big|\mathrm{D}_x^2 f(x_c, a_1)\Big| \prod\limits_{l=2}^{t} \Big|\mathrm{D}_x f(x_l, a_l)\Big| \le |\Delta x_1| L S_x^{t-1} < 1$. Hence, $|\Delta x_1| \le \delta_x = 1/(LS_x^{t-1})$.

Therefore, if the perturbation Δx_1 is less than the value δ_x, then the periodic orbit is stable. However, at parametric perturbations the maximum of Δx_1 is given by the inequality (15). Thus, finally we get: $\delta_a t S_a \sum_{i=1}^{t} S_x^i = 1 \Big/ \left(LS_x^{t-1}\right)$ or $\delta_a = 1 \Big/ \left(t S_a L S_x^{t-1} \sum\limits_{i=1}^{t} S_x^i\right)$. □

For the quadratic family of maps at $a \in [0, 4]$ and $x \in [1/4, 3/4]$ we find: $S_a = \max\limits_{x,a} \left|\frac{\partial}{\partial a} f(x, a)\right| = \frac{1}{4}$, $S_x = \max\limits_{x,a} \left|\frac{\partial}{\partial x} f(x, a)\right| = 2$, $L = \max\limits_{x,a} \left|\frac{\partial^2}{\partial x^2} f(x, a)\right| = 8$. This gives a sufficient estimate of the admissible errors in the parameter values: $\delta_a \le 1 \Big/ \left(t 2^t \sum_{i=1}^{t} 2^i\right)$, $\delta_x \le 1/2^{t+2}$.

From the obtained results we may get the following important consequence. If the induced periodic dynamics in a polymodal family under a periodic perturbation is observed, then it can not be destroyed by a small enough external noise which smears the parameter values $\hat{a}$. The maximal noise level can be estimated by the above theorem 2.

3.2. **A piecewise linear family of maps.** Consider the following family of maps of the interval $[0,1]$ into itself:

$$T_a:\ x \longmapsto f(x,a)\ =\ \begin{cases} q(a)x + r(a)\ , 0 \le x \le a, \\ \\ p(a)(x-1)\ , a < x \le 1, \end{cases} \tag{16}$$

where $a \in (0,1)$ is a control parameter and $q(a) = (1-a)/(a(2-a))$, $r(a) = 1/(2-a)$, $p(a) = -1/(1-a)$. All maps of the family (16) are constructed in such a way that the critical point $x_c = a$ hits in the unstable fixed point $r(a) \equiv \tilde{x}$ after three iterations (Fig.4). First, let us prove that these maps has a mixing attractor for an arbitrary $a \in (0,1)$, i.e. they are chaotic.

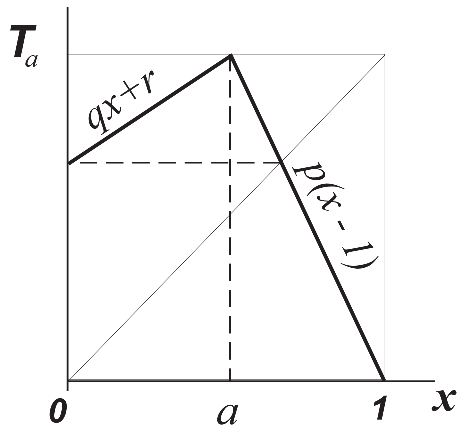

FIGURE 4. A piecewise linear map determined by the family (16).

Let T be a map of a set M into itself and $\Lambda \subset M$ be a compact invariant subset which is different from a periodic orbit. We say (see [28, 29]) that Λ is a mixing set if for any open set U in Λ and any finite covering $\Sigma = \{\sigma_j\}$ of the set Λ there exist $m = m(U,\Sigma)$ and $r \ge 1$ depending only on Λ such that $T^m\left(\bigcup_{i=0}^{r-1} T^i U\right) \bigcap \sigma_j \ne \emptyset$ for all j. If, in addition, the set Λ attracts almost all trajectories from a neighborhood V, i.e. there exists $V \supset \Lambda$ such that $V \ne \Lambda$, $TV \subset V$ and $\bigcap_{i>0} T^i V = \Lambda$, then the set Λ is said to be a mixing attractor.

The existence of the mixing attractor in maps of an interval into itself is a sufficiently strong property. For example, a map with the mixing attractor does not have stable periodic orbits, and it possesses sensitive dependence on initial conditions. Moreover, for such maps it is possible to construct absolutely continuous invariant measure. Thus, maps with the mixing attractor can be called chaotic. Moreover, the following result holds [28]: if $T : x \mapsto f(x)$, $x \in I$, and $f \in C^0(I,I)$ where I is an interval, then the mixing attractor consists of one or several intervals which are cyclically mapped into each other, and periodic points are dense on it.

Now we may get the chaoticity conditions in the family (17).

Theorem 3. *For any $a \in (0,1)$ the map T_a (16) has the mixing attractor $\Lambda = [0,1]$.*

Proof. We make it by two stages.

1) To show that $\Lambda = [0,1]$ is an attractor for T_a, it is sufficient to consider the map (16) in any extended interval, for example, in $[-1/2,\ 3/2]$ (Fig.5). One can see that for any $a \in (0,1)$ there exists a neighborhood V of the set Λ such that $fV \subset V$ and $f^2V = \Lambda$. Thus, Λ is the attractor on the interval $[0,1]$.

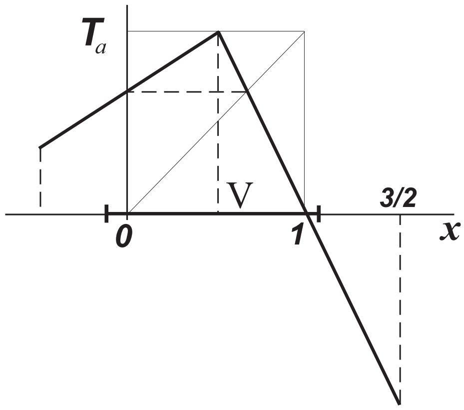

FIGURE 5. An extended piecewise linear map (16).

2) To prove that the attractor $\Lambda = [0,1]$ is a mixing, note that it consists of two subintervals, J_1 and J_2, which are mapped into each other under the action of T_a, $\Lambda = J_1 \bigcup J_2$, $T_aJ_1 = J_2$, $T_aJ_2 = J_1$, where $J_1 = [0,\tilde{x}]$, $J_2 = [\tilde{x},1]$ (Fig.6). The rest of proof uses the following result.

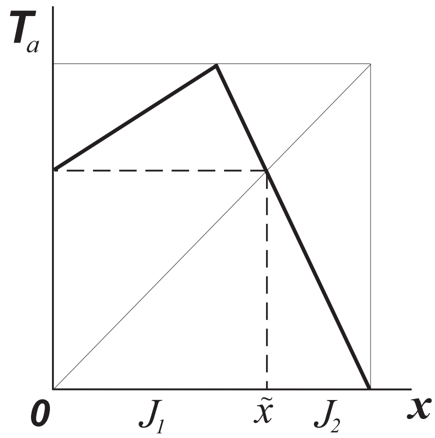

FIGURE 6. The intervals J_1, J_2 composed the attractor $\Lambda = [0,1]$ for the map (16).

Lemma 1. *For any open set $U \subset \Lambda$ there exists a number n such that*
i) $T_a^nU = \Lambda$ or
ii) $T_a^nU = J_i$, where $i = 1,2$.

Proof. Consider the map $T_a^2 = T_a \circ T_a$ (Fig.7). Because $T_a^2 J_1 = J_1$ and $T_a^2 J_2 = J_2$, then T_a^2 is decomposed into two maps, T_{J_1} and T_{J_2}, operating on J_1 and J_2, respectively. Moreover, owing to $p^2 = p(a)p(a) > 1$, $|p(a)q(a)| = 1/(a(2-a)) > 1$, $a \in (0,1)$, the maps T_{J_1}, T_{J_2} are expanding ones. Therefore, for any open set $U \subset J_1$ ($U \subset J_2$) there exists a number m such that $T_a^{2m}U = J_1$ ($T_a^{2m}U = J_2$). In other words, for $n = 2m$ we have the condition ii). Now it is necessary to consider the sets $U \ni \tilde{x}$. For such sets (Fig.7) there exists $m : T_a^{2m}U = \Lambda$. □

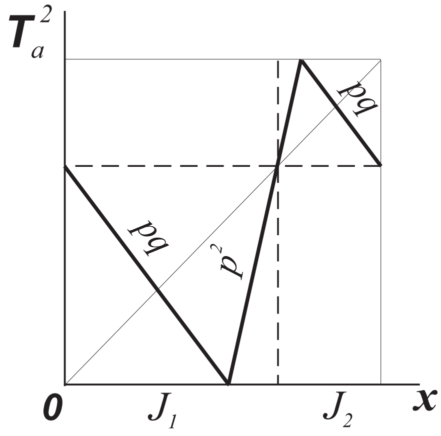

Figure 7. The second iteration, T_a^2, of the map T_a (16).

Thus, to prove theorem 3 it is necessary, for any open set $U \subset \Lambda$, to choose $r = 2$ and m from the above lemma. In this case $T_a^m(T_aU \bigcup U) = \Lambda$ because, if $U \subset J_1$ then $T_aU \subset J_2$, and if $T_a^mU = J_1$ then $T_a^m(T_aU) = J_2$. By the same manner we obtain the result for $U \subset J_2$ and $U \ni \tilde{x}$. □

It is not hard to understand the structure of periodic orbits of the family (16). Because T_a^2 is decomposed into two independent maps, T_{J_1} and T_{J_2} which operate in the subintervals J_1 and J_2 respectively, it makes it possible to construct maps having orbits of an arbitrary even period, T_a^{2k}. These orbits are dense in $\Lambda = [0,1]$. For this it is sufficient to find $T_{J_1}^k$ and $T_{J_2}^k$.

A completely different situation takes place for odd iterations of T_a. Really, $T_a^{2k+1} = T_a \circ T_a^{2k}$. Therefore, except for a fixed point, T_a does not have orbits of odd periods. Thus, the family (16) has the mixing attractor with dense periodic orbits of even periods.

Now let us consider the perturbed family (16). Restrict ourselves by the case of a 2-periodic transformation of the parameter a:

$$\begin{cases} T_1 : x \longmapsto F_1(x) \equiv T_{a_2} \circ T_{a_1}, \\ \\ T_2 : x \longmapsto F_2(x) \equiv T_{a_1} \circ T_{a_2}. \end{cases} \tag{17}$$

Without loss of generality suppose that $0 < a_1 < a_2 < 1$. Introduce the following notations: $a_1 = a$, $a_2 = a + \varepsilon$, $\varepsilon > 0$ (Fig.8). Thus, the map T_1 has three fixed points which exist for any $a_1, a_2 \in (0,1)$. These fixed points correspond to three different 2-periodic orbits of the perturbed map (17). The orbit corresponding to the middle fixed point (Fig.8) arises from the fixed point of the unperturbed map (17). Two orbits of period 2 corresponding to two other fixed points arise from an orbit of period 2 (Fig.7).

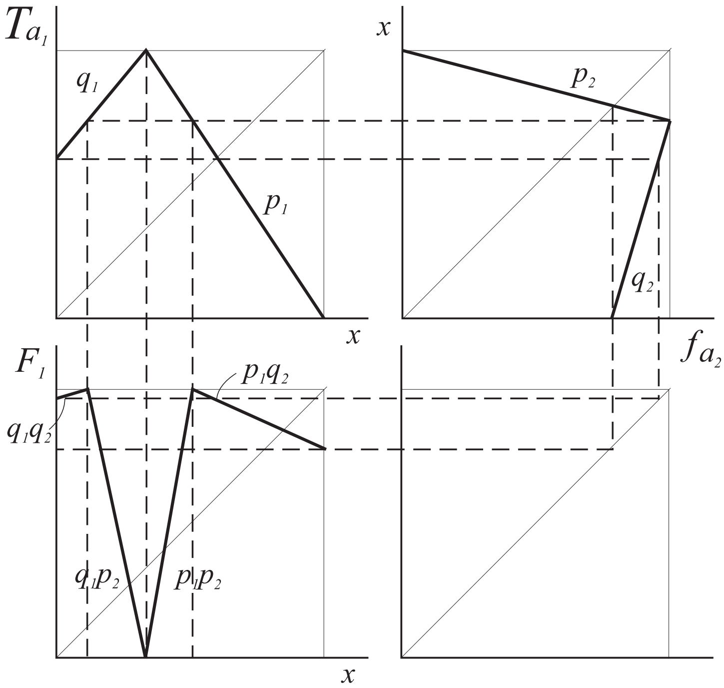

FIGURE 8. Construction of the map T_1 (17).

Let us show that it is possible to find such parameter values that the latter fixed points become stable. We have: $|q_1p_2| = (1-a)/(a(2-a)(1-a-\varepsilon)) \equiv s_1(\varepsilon)$, $|q_2p_1| = (1-a-\varepsilon)/\big((a+\varepsilon)(2-a-\varepsilon)(1-\varepsilon)\big) \equiv s_2(\varepsilon)$. Consider the functions $s_1(\varepsilon)$, $s_2(\varepsilon)$ in the region $0<\varepsilon<1-a$ (Fig.9). Thus, we find that for any $a \in (0,1)$ there is a range of the values $\varepsilon \in (\varepsilon^*, 1-a)$ where $s_2(\varepsilon) < 1$. In other words, in the interval $(\varepsilon^*, 1-a)$ the perturbed map (17) has the stabilized periodic orbit arisen from the unstable orbit of the unperturbed map (16) (see Figs.7,8), and almost all phase points from the interval $[0,1]$ are attracted to it. This orbit becomes stable by means of continuous change in the parameters (a_1, a_1) of the map (17) to the values (a_1, a_2) such that $s_2(a_1, a_2) < 1$ (see Fig.9).

In view of the this obtained result the following question arises: Is it possible to stabilize a prescribed unstable orbit in the chaotic map (16) via the feedback–free perturbation (2)?

Our preliminary considerations show that this problem requires detailed analysis of the map (16) and application of specified perturbations.

3.3. **Two–dimensional map with a hyperbolic attractor.** External perturbations can crucially affect on dynamics of certain maps if their dimension is more than one. As an example of such a map let us consider so-called Belykh map which appears in the physical systems of phase synchronization [30].

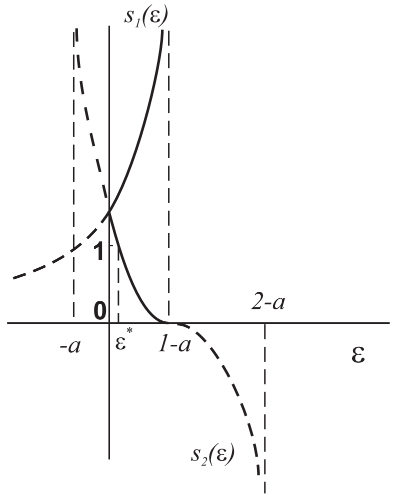

FIGURE 9. The functions $s_1(\varepsilon),\ s_2(\varepsilon)$ for $a = 1/2$.

Let $Q = \{(x, y) :\ |x| < 1,\ |y| < 1\}$ be a square in the plane (x, y). Then the Belykh map is given by the following construction [31]:

$$T :\ (x, y) \longmapsto f(x, y), \tag{18}$$

where

$$f(x,y) = \begin{cases} \left(\lambda_1(x+1) - 1,\ \dfrac{1}{\lambda_2}(y+1) - 1\right), (x, y) \in Q_1, \\ \left(\lambda_3(x-1) + 1,\ \dfrac{1}{\lambda_4}(y-1) + 1\right), (x, y) \in Q_2, \end{cases} \tag{19}$$

and the regions Q_1, Q_2 are presented as the separation of the function $h(x)$: $[-1, 1] \to [-1, 1]$ into two parts:

$$\begin{aligned} Q_1 &= \Big\{(x, y) \in Q :\ y < h(x)\Big\}\ , \\ Q_2 &= \Big\{(x, y) \in Q :\ y > h(x)\Big\}\ . \end{aligned} \tag{20}$$

In addition, the constants $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$ and the function $h(x)$ should be chosen in such a way that the square Q is mapped into itself, $TQ \subset Q$ (Fig.10).

We consider the case $h(x) = ax,\ \lambda_1 = \lambda_3,\ 1/\lambda_2 = 1/\lambda_4 \equiv \lambda_2$, i.e.

$$T :\ (x, y) \longmapsto f(x, y) = \begin{cases} \Big(\lambda_1(x+1) - 1,\ \lambda_2(y+1) - 1\Big)\ ,\quad y < ax, \\ \Big(\lambda_1(x-1) + 1,\ \lambda_2(y-1) + 1\Big)\ ,\quad y > ax, \end{cases} \tag{21}$$

$|a| < 1$ (Fig.11). The Belykh map is remarkable for the fact that it has a hyperbolic attractor for certain parameter values.

As is known (see, e.g., [29, 32]), an invariant set Λ for a diffeomorphism $f : Q \to Q$ of a compact manifold Q is said to be hyperbolic attractor if Λ is an attractor and simultaneously hyperbolic set (a rigorous definition can be found in [32]). Let us clarify the sense of the notion of the hyperbolic attractor. If the set Λ is an attractor then there is an open neighborhood which shrinks to Λ with iterations.

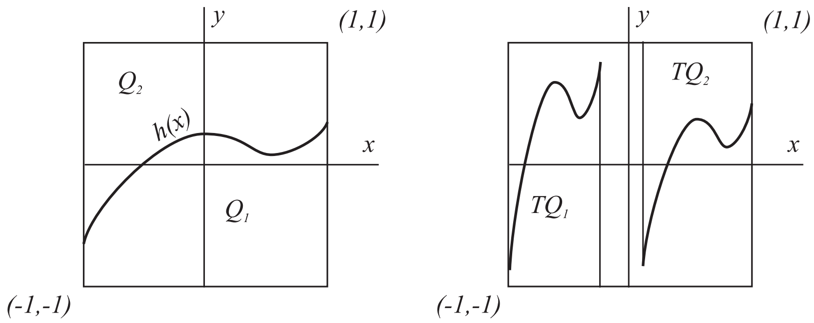

FIGURE 10. Construction of the Belykh map (18)-(20).

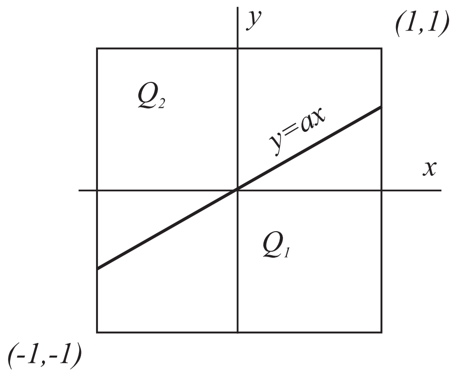

FIGURE 11. A special case (21) of the Belykh map (18)–(20).

For maps, the property of hyperbolicity means that in any point p of Λ there are two invariant directions. Along one of them the points of the manifold Q exponentially tend to the initial point p at $t \to \infty$, and along the other direction the points tend to p at $t \to -\infty$. In turn, the existence of stable and unstable manifolds implies that maps with hyperbolic attractors have sensitive dependence on initial conditions. Moreover, such maps possess invariant measures which determine statistical properties of typical trajectories.

We consider the Belykh map (21) which is however not hyperbolic in a rigorous sense, because it has discontinuities. But this map is typified by hyperbolic dynamical systems with singularities. Such maps appear in many physical problems. With the proviso that a set of discontinuities has a zero measure and the other additional conditions [33], one can obtain the rigorous results concerning discontinuous hyperbolic dynamical systems. In particular, for every regular point it is possible to form the local stable and unstable manifolds. Moreover, based on the concrete type of points of discontinuity one can construct an ergodic invariant measure.

It is not difficult to find conditions at which there exists the hyperbolic attractor in the Belykh map. First, let us note that for $|a| < 1$ the map (21) has two fixed points $X = (1, 1)$ and $Y = (-1, -1)$. Second, for all points of the square where the map (21) is determined, the derivative is $Df = \text{diag}\{\lambda_1, \lambda_2\}$. For the property of hyperbolicity it is necessary that $|\lambda_1| < 1$, $|\lambda_2| > 1$ or vice versa, and $TQ \subset Q$. It is not hard to verify that the latter condition is satisfied only if $0 < \lambda_1 < 1$, $0 < \lambda_2 < 2/(1+|a|)$, $|a| < 1$. Finally, the transformation T should be a one-to-one map. Geometrically, we may come to the conclusion that the map (21) is a homeomorphism only in the parameter range $0 < \lambda_1 < 1/2$. Thus, for a hyperbolicity we get the following inequalities: 1) $0 < \lambda_1 < 1/2$; 2) $1 < \lambda_2 < 2/(1+|a|)$; and 3) $|a| < 1$.

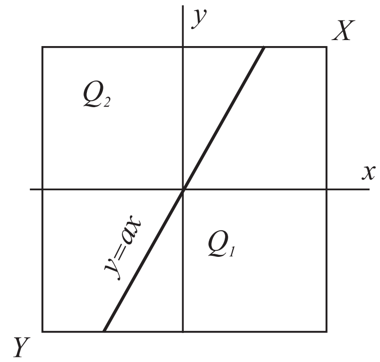

FIGURE 12. The generalized Belykh map (22).

Let us generalize the Belykh map (21) into the case of $|a| > 1$. To this end one should note that for $|a| > 1$ the point X falls in the range $y < ax$, and the point Y falls in the region $y > ax$ (Fig.12). Therefore, to determine for $|a| > 1$ the existence conditions of these fixed points it is necessary to rewrite the map (21) in the form:

$$T:\ (x,y) \longmapsto f(x,y) = \begin{cases} (\lambda_1(x+1)-1,\ \lambda_2(y+1)-1), & y > ax, \\ \\ (\lambda_1(x-1)+1,\ \lambda_2(y-1)+1), & y < ax. \end{cases} \tag{22}$$

Thus, we can obtain the map (22) from the map (21) by the substitution $x \leftrightarrow y$ and $a \to 1/a$. Consequently, we get that the hyperbolicity condition for the map (22) is the validity of the following inequalities $0 < \lambda_2 < 1/2$ and $1 < \lambda_1 < 2/(1+1/|a|)$. It should be however noted that in contrast to the map (21), in this case $|\lambda_2| < 1$ and $|\lambda_1| > 1$.

Let us consider a construction of the Belykh map for the case of a two–periodically perturbed parameter a. To find qualitative changes in its dynamics it is necessary to switch the parameter a near $a = 1$ in such a way that $a_1 < 1$, $a_2 > 1$. In order that for both these cases the perturbed map would be hyperbolic it is necessary to vary also the parameter λ_1, λ_2. Taking into account these conditions, we may write the perturbed Belykh map as follows:

$$\bar{T} \ = \begin{cases} (x,y) \longmapsto f(a_2,\lambda_1^2,\lambda_2^2) \circ f(a_1,\lambda_1^1,\lambda_2^1)(x,y) \\ \\ (x,y) \longmapsto f(a_1,\lambda_1^1,\lambda_2^1) \circ f(a_2,\lambda_1^2,\lambda_2^2)(x,y) \end{cases} \tag{23}$$

for even and odd iterations, respectively.

Because both maps (with $a_1 < 1$ and $a_2 > 1$) have the fixed points $X = (1,1)$ and $Y = (-1,-1)$, then these points remain to be the fixed ones also for the perturbed map (23). Moreover, the differential $D\bar{T}$ of the perturbed map (for odd and even iterations) is:

$$D\bar{T} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1^1 & 0 \\ 0 & \lambda_2^1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1^2\lambda_1^1 & 0 \\ 0 & \lambda_2^2\lambda_2^1 \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} \lambda_1^* & 0 \\ 0 & \lambda_2^* \end{pmatrix}.$$

Therefore, in view of the fact that for $a_1 < 1$ we have $0 < \lambda_1^1 < 1/2$, $1 < \lambda_2^1 < 2/(1+|a_1|)$, and $1 < \lambda_1^2 < 1/(1+1/|a_2|)$, $0 < \lambda_2^2 < 1/2$ for $a_2 > 1$, the eigenvalues λ_1^* and λ_2^* of $D\bar{T}$ are changed in the following range: $0 < \lambda_1^* < 1/(1+1/|a_2|)$, $0 < \lambda_2^* < 1/(1+|a_1|)$. In other words, we find that $|\lambda_1^*| < 1$, $|\lambda_2^*| < 1$ and the fixed points X, Y of the map become stable. This means that the hyperbolic attractor is degenerated and replaced with a simple attractor.

Thus, introducing periodic perturbations of the map with the hyperbolic attractor we get a qualitative change in the dynamics: From a chaotic map it transforms into a regular one with the stable fixed points.

4. **Conclusion.** Thus, we have shown that external periodic perturbations can crucially effect on the behavior of the quadratic map, a piecewise linear family and a map with the hyperbolic attractor. Moreover, as shown in Section 3.1, for maps having critical points the chosen in advance periodic orbits can be stabilized. Hence, for dynamical systems the behavior of which is effectively described by polymodal one–dimensional maps, the non–feedback control is possible. Therefore, in general, the obtained results allows us to put a question concerning a rigorous validation of the existence of a feedback–free parametric excitation needed for the stabilization of the prescribed periodic orbits embedded in a chaotic attractor.

In addition, the developed technique allows, in principle, to find analytically a way for the chaos suppression phenomenon for chaotic maps with external perturbations. This fact can essentially simplify investigations of the maps under periodic perturbations.

Thus, the described results permit us to find an analytic approach to the problem of the chaos suppression for dynamical systems with continuous time (i.e. for flows). Let us suppose that a system possesses a chaotic attractor. Then, if we appropriately choose external periodic perturbations then one can expect that they lead to appearance of stable periodic orbits. These orbits either have not existed in the initial (unperturbed) system or they have not been stable ones. Some investigations justify this conjecture (see [2] and references therein). However the main problem is to show the presence of an appropriate chaotic attractor in the system. At the same time, some results concerning the induced stable periodic behavior in continuous non–chaotic dynamical systems possessing only steady state and/or unstable cycles have been described [2].

On the other hand, great successes of the chaos suppression phenomena in applications led to the opinion that chaos can always be suppressed by external perturbations. However this is not the case. It has been found that periodic perturbations of certain parameters can not lead to the suppression of chaos. As a consequence, for an arbitrary system we do not know in advance, what parameter is appropriate for the stabilization of the system dynamics [34]. In this connection, it would be useful to have a basic criterion which allows us to determine in what cases the chaotic motion can be stabilized. This important question will arise every time in practice. But unfortunately, up to the present there is no even general analytical criteria of the existence of chaos in simple dynamical systems.

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